Notes on the function gsw_thermobaric(SA, CT, p)

Notes written 15th May 2015

This function, **gsw_thermobaric** calculates the thermobaric coefficient according to Eqn. (3.8.2) of the TEOS-10 manual (IOC *et al*. (2010)), namely

$$T_{\rm b}^{\Theta} = T_{\rm b}^{\Theta}\left(S_{\rm A},t\,p\right) = \beta^{\Theta}\,\frac{\partial\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\partial P}\bigg|_{S_{\rm A},\Theta} = \frac{\partial\alpha^{\Theta}}{\partial P}\bigg|_{S_{\rm A},\Theta} - \frac{\alpha^{\Theta}}{\beta^{\Theta}}\,\frac{\partial\beta^{\Theta}}{\partial P}\bigg|_{S_{\rm A},\Theta}. \qquad (3.8.2)$$

The input variables are Absolute Salinity $S_{\rm A}$, Conservative Temperature, and pressure. This function uses the 75-term polynomial function expression for specific volume **gsw_specvol**(SA,CT,p). This 75-term polynomial expression for specific volume is discussed in Roquert *et al*. (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al*. (2010)).For dynamical oceanography we may take the 75-term polynomial expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

A discussion of the thermobaric coefficient and the thermobaric process whereby epineutral diffusion causes dianeutral advection may be found in section 3.8 and appendix A.14 of the TEOS-10 manual, and these sections are repeated below.

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. *Ocean Modelling*, **90**, pp. 29-43. http://dx.doi.org/10.1016/j.ocemod.2015.04.002

Here follows sections 3.8, 3.11, 3.13 and appendix A.14 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.8 Thermobaric coefficient

The thermobaric coefficient quantifies the rate of variation with pressure of the ratio of the thermal expansion coefficient and the saline contraction coefficient. With respect to potential temperature θ the thermobaric coefficient is (McDougall (1987b))

$$T_{\rm b}^{\theta} = T_{\rm b}^{\theta}\left(S_{\rm A}, t, p\right) = \beta^{\theta} \left.\frac{\partial\left(\alpha^{\theta}/\beta^{\theta}\right)}{\partial P}\right|_{S_{\rm A},\theta} = \left.\frac{\partial\alpha^{\theta}}{\partial P}\right|_{S_{\rm A},\theta} - \frac{\alpha^{\theta}}{\beta^{\theta}}\left.\frac{\partial\beta^{\theta}}{\partial P}\right|_{S_{\rm A},\theta}. \tag{3.8.1}$$

This expression for the thermobaric coefficient is most readily evaluated by differentiating an expression for density expressed as a function of potential temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \tilde{\rho}\left(S_{\rm A}, \theta, p\right)$.

With respect to Conservative Temperature Θ the thermobaric coefficient is

$$T_{\rm b}^{\Theta} = T_{\rm b}^{\Theta}\left(S_{\rm A}, t, p\right) = \beta^{\Theta} \left.\frac{\partial\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\partial P}\right|_{S_{\rm A},\Theta} = \left.\frac{\partial\alpha^{\Theta}}{\partial P}\right|_{S_{\rm A},\Theta} - \frac{\alpha^{\Theta}}{\beta^{\Theta}}\left.\frac{\partial\beta^{\Theta}}{\partial P}\right|_{S_{\rm A},\Theta}. \tag{3.8.2}$$

This expression for the thermobaric coefficient is most readily evaluated by differentiating an expression for density expressed as a function of Conservative Temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \hat{\rho}\left(S_{\rm A}, \Theta, p\right)$.

The thermobaric coefficient enters various quantities to do with the path-dependent nature of neutral trajectories and the ill-defined nature of neutral surfaces (see (3.13.1) – (3.13.7)). The thermobaric dianeutral advection associated with the lateral mixing of heat and salt along neutral tangent planes is given by $e^{\rm Tb} = -gN^{-2}K\,T_{\rm b}^{\theta}\nabla_n\theta\cdot\nabla_n P$ or $e^{\rm Tb} = -gN^{-2}K\,T_{\rm b}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$ where $\nabla_n\theta$ and $\nabla_n\Theta$ are the two-dimensional gradients of either potential temperature or Conservative Temperature along the neutral tangent plane, $\nabla_n P$ is the corresponding epineutral gradient of Absolute Pressure and K is the epineutral diffusion coefficient. Note that the thermobaric dianeutral advection is proportional to the mesoscale eddy flux of "heat" along the neutral tangent plane, $-c_p^0 K\nabla_n\Theta$, and is independent of the amount of small-scale (dianeutral) turbulent mixing and hence is also independent of the dissipation of mechanical energy ε (Klocker and McDougall (2010a)). It is shown in appendix A.14 below that while the epineutral diffusive fluxes $-K\nabla_n\theta$ and $-K\nabla_n\Theta$ are different, the product of these fluxes with their respective thermobaric coefficients is the same, that is, $T_{\rm b}^{\theta}\nabla_n\theta = T_{\rm b}^{\Theta}\nabla_n\Theta$. Hence the thermobaric dianeutral advection $e^{\rm Tb}$ is the same whether it is calculated as $-gN^{-2}K\,T_{\rm b}^{\theta}\nabla_n\theta\cdot\nabla_n P$ or as $-gN^{-2}K\,T_{\rm b}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$. Expressions for $T_{\rm b}^{\theta}$ and $T_{\rm b}^{\Theta}$ in terms of enthalpy in the functional forms $\tilde{h}\left(S_{\rm A}, \theta, p\right)$ and $\hat{h}\left(S_{\rm A}, \Theta, p\right)$ can be found in appendix P.

Interestingly, for given magnitudes of the epineutral gradients of pressure and Conservative Temperature, the dianeutral advection, $e^{\rm Tb} = -gN^{-2}K\,T_{\rm b}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$, of thermobaricity is maximized when these gradients are parallel, while neutral helicity is maximized when these gradients are perpendicular, since neutral helicity is proportional to $T_{\rm b}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k}$ (see Eqn. (3.13.2)).

This thermobaric vertical advection process, $e^{\rm Tb}$, is absent from standard layered ocean models in which the vertical coordinate is a function only of $S_{\rm A}$ and Θ (such as σ_2, potential density referenced to 2000 dbar). As described in appendix A.27 below, the isopycnal diffusion of heat and salt in these layered models, caused by both parameterized diffusion along the coordinate and by eddy-resolved motions, does give rise to the cabbeling advection through the coordinate surfaces but does not allow the thermobaric velocity $e^{\rm Tb}$ through these surfaces (Klocker and McDougall (2010a)).

In both the SIA and GSW computer software libraries the thermobaric parameter is output in units of $\rm K^{-1}\ Pa^{-1}$.

3.11 Neutral tangent plane

The neutral plane is that plane in space in which the local parcel of seawater can be moved an infinitesimal distance without being subject to a vertical buoyant restoring force; it is the plane of neutral- or zero- buoyancy. The normal vector to the neutral tangent plane $\mathbf{n}$ is given by

$$\begin{aligned} g^{-1} N^2 \mathbf{n} &= -\rho^{-1}\nabla\rho + \kappa\nabla P \;=\; -\rho^{-1}\left(\nabla\rho - \nabla P / c^2\right) \\ &= \alpha^{\theta}\nabla\theta - \beta^{\theta}\,\nabla S_{\mathrm{A}} \\ &= \alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}. \end{aligned} \qquad (3.11.1)$$

As defined, $\mathbf{n}$ is not quite a unit normal vector, rather its vertical component is exactly $\mathbf{k}$, that is, its vertical component is unity. It is clear that $\alpha^{\theta}\nabla\theta - \beta^{\theta}\,\nabla S_{\mathrm{A}}$ is exactly equal to $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}$. Interestingly, both $\alpha^{\theta}\nabla\theta$ and $\beta^{\theta}\,\nabla S_{\mathrm{A}}$ are independent of the four arbitrary constants of the Gibbs function (see Eqn. (2.6.2)) while both $\alpha^{\Theta}\nabla\Theta$ and $\beta^{\Theta}\,\nabla S_{\mathrm{A}}$ contain an identical additional arbitrary term proportional to $a_3\,\nabla S_{\mathrm{A}}$; terms that exactly cancel in their difference, $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}$, in Eqn. (3.11.1).

Expressing the two-dimensional gradient of properties in the neutral tangent plane by ∇_n, the property gradients in a neutral tangent plane obey

$$\begin{aligned} -\rho^{-1}\nabla_n\rho + \kappa\nabla_n P \;=\; -\rho^{-1}\left(\nabla_n\rho - \nabla_n P / c^2\right) &= \alpha^{\theta}\nabla_n\theta - \beta^{\theta}\,\nabla_n S_{\mathrm{A}} \\ &= \alpha^{\Theta}\nabla_n\Theta - \beta^{\Theta}\,\nabla_n S_{\mathrm{A}} \\ &= \mathbf{0}. \end{aligned} \qquad (3.11.2)$$

Here ∇_n is an example of a projected gradient

$$\nabla_r\tau \equiv \left.\tfrac{\partial\tau}{\partial x}\right|_r \mathbf{i} + \left.\tfrac{\partial\tau}{\partial y}\right|_r \mathbf{j} + 0\mathbf{k}, \qquad (3.11.3)$$

that is widely used in oceanic and atmospheric theory and modelling. Horizontal distances are measured between the vertical planes of constant latitude x and longitude y while the values of the property τ are evaluated on the r surface (e. g. an isopycnal surface, or in the case of ∇_n, a neutral tangent plane). This coordinate system is described by Sutcliffe (1947), Bleck (1978), McDougall (1987b), McDougall (1995) and Griffies (2004). Note that $\nabla_r\tau$ has no vertical component; it is not directed along the r surface, but rather it points in exactly the horizontal direction.

Finite difference versions of Eqn. (3.11.2) such as $\bar{\alpha}^{\Theta}\Delta\Theta - \bar{\beta}^{\Theta}\,\Delta S_{\mathrm{A}} \approx 0$ are also very accurate. Here $\bar{\alpha}^{\Theta}$ and $\bar{\beta}^{\Theta}$ are the values of these coefficients evaluated at the average values of Θ, S_{A} and p of two parcels $\left(S_{\mathrm{A}}^1, \Theta_1, p_1\right)$ and $\left(S_{\mathrm{A}}^2, \Theta_2, p_2\right)$ on a "neutral surface" and $\Delta\Theta$ and ΔS_{A} are the property differences between the two parcels. The error involved with this finite amplitude version of Eqn. (3.11.2), namely

$$-\,T_{\mathrm{b}}^{\Theta}\int_1^2 \left(P - \bar{P}\right)\,d\Theta, \qquad (3.11.4)$$

is described in section 2 and appendix A(c) of Jackett and McDougall (1997). An equally accurate finite amplitude version of Eqn. (3.11.2) is to equate the potential densities of the two fluid parcels, each referenced to the average pressure $\bar{p} = 0.5\left(p_1 + p_2\right)$.

The reason why oceanographers take the strong lateral mixing of mesoscale eddies to be directed along the neutral tangent plane is because of the smallness of the observed dissipation of mechanical energy ε in the ocean interior. If the lateral diffusivity $K \approx 10^2 - 10^3\,\mathrm{m^2\,s^{-1}}$ of mesoscale dispersion and subsequent molecular diffusion were to occur along a surface that differed in slope from the neutral tangent plane by an angle whose tangent was s, then the individual fluid parcels would be transported above and below the neutral tangent plane and would need to subsequently sink or rise in order to attain a vertical position of neutral buoyancy.

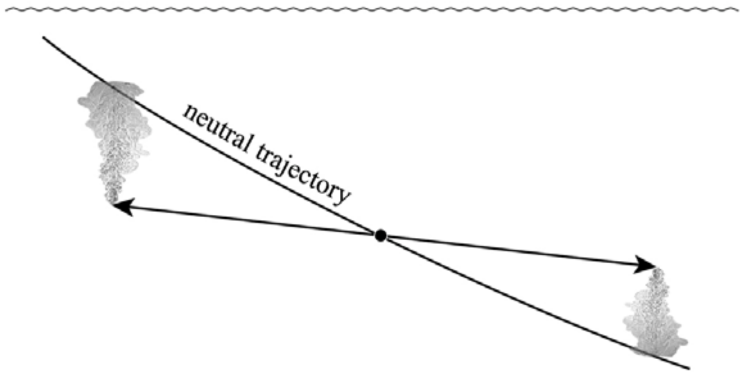

Figure 5. Sketch of the consequences of the adiabatic movement followed by release of fluid parcels along a plane that is different to a neutral tangent plane.

This vertical motion would either (i) involve no small-scale turbulent mixing, in which case the combined process is equivalent to epineutral mixing, or (ii), the sinking and rising parcels would mix with and entrain the surrounding ocean in a plume-like fashion (see Figure 5), so suffering irreversible diffusion. In this second case, the dissipation of mechanical energy associated with the diapycnal mixing would be observed. But in fact the dissipation of mechanical energy in the main thermocline is consistent with a diapycnal diffusivity of only $10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. This small value of the diapycnal (vertical) diffusivity has been confirmed by purposely released tracer experiments.

When lateral diffusion with diffusivity K is taken to occur along a surface other than a neutral tangent plane, some dianeutral diffusion occurs, and the amount of this dianeutral diffusion is the same as achieved by a vertical diffusivity of s^2K where s^2 is the square of the vector slope $\nabla_r z - \nabla_n z$ between the mixing direction and the neutral tangent plane. This result is proven as follows.

The lateral flux of Neutral Density along the direction of mixing, the r surface is

$$-K\nabla_r \gamma \;=\; -K\gamma_z\left(\nabla_r z - \nabla_n z\right) , \qquad (3.11.5)$$

and the component of this lateral flux across the neutral tangent plane is

$$-K\nabla_r \gamma \cdot \left(\nabla_r z - \nabla_n z\right) \;=\; -K\gamma_z\left(\nabla_r z - \nabla_n z\right)^2 . \qquad (3.11.6)$$

Dividing by minus the vertical gradient of Neutral Density, $-\gamma_z$, shows that this flux is the same as that caused by the positive fictitious vertical diffusivity of density $\left(\nabla_r z - \nabla_n z\right)^2 K \;= s^2K$.

Hence if all of this observed diapycnal diffusivity (based on the observed dissipation of turbulent kinetic energy ε) were due to mesoscale eddies mixing along a direction different to neutral tangent planes, the (tangent of the) angle between this mesoscale mixing direction and the neutral tangent plane, s, would satisfy $10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-1} = s^2 K$. Using $K \approx 10^3\,\mathrm{m}^2\,\mathrm{s}^{-1}$ gives the maximum value of s to be 10^{-4}. Since we believe that bona fide interior diapycnal mixing processes (such as breaking internal gravity waves) are responsible for the bulk of the observed diapycnal diffusivity, we conclude that the angular difference s between the direction of mesoscale eddy mixing and the neutral tangent plane must be substantially less than 10^{-4}; say $2x10^{-5}$ for argument's sake.

3.13 Neutral helicity

The neutral tangent plane was defined in section 3.11 as the plane in which parcels can be moved in an adiabatic and isohaline manner without experiencing a vertical buoyant force. The normal $\mathbf{n}$ to the neutral tangent plane is given by Eqn. (3.11.1) and it is natural to think that all these little tangent planes would link up and form a well-defined surface, but this is not actually the case in the ocean. In order to understand why the ocean chooses to be so ornery we need to understand what property the normal $\mathbf{n}$ to a surface must fulfill in order that the surface exists.

In general, for a surface to exist in (x,y,z) space there must be a function $\phi(x,y,z)$ that is constant on the surface and whose gradient $\nabla\phi$ is in the direction of the normal to the surface, $\mathbf{n}$. That is, there must be an integrating factor $b(x,y,z)$ such that $\nabla\phi = b\mathbf{n}$. Assuming now that the surface does exist, consider a line integral of $b\mathbf{n}$ along a closed curved path in the surface. Since the line element of the integration path is everywhere normal to $\mathbf{n}$, the closed line integral is zero, and by Stokes's theorem, the area integral of $\nabla\times\left(b\mathbf{n}\right)$ must be zero over the area enclosed by the closed curved path. Since the area element of integration $d\mathbf{A}$ is in the direction $\mathbf{n}$, it is clear that $\nabla\times\left(b\mathbf{n}\right)\cdot d\mathbf{A}$ is proportional to $\nabla\times\left(b\mathbf{n}\right)\cdot\mathbf{n}$. The only way that this area integral can be guaranteed to be zero for all such closed paths is if the integrand is zero everywhere on the surface, that is, if $\nabla\times\left(b\mathbf{n}\right)\cdot\mathbf{n} = \left(\nabla b\times\mathbf{n}\right)\cdot\mathbf{n} + b\left(\nabla\times\mathbf{n}\right)\cdot\mathbf{n} = 0$, that is, if $\mathbf{n}\cdot\nabla\times\mathbf{n} = 0$ at all locations on the surface.

For the case in hand, the normal to the neutral tangent plane is in the direction $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}$ and we define the neutral helicity H^{n} as the scalar product of $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}$ with its curl,

$$H^{\mathrm{n}} \equiv \left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}\right)\cdot\nabla\times\left(\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}\right). \tag{3.13.1}$$

Neutral tangent planes (which do exist) do not link up in space to form a well-defined neutral surface unless the neutral helicity H^{n} is everywhere zero on the surface.

Recognizing that both the thermal expansion coefficient and the saline contraction coefficient are functions of $\left(S_{\mathrm{A}},\Theta,p\right)$, neutral helicity H^{n} may be expressed as the following four expressions, all of which are proportional to the thermobaric coefficient T_{b}^{Θ} of the equation of state,

$$\begin{aligned} H^{\mathrm{n}} &= \beta^{\Theta} T_{\mathrm{b}}^{\Theta}\,\nabla P\cdot\nabla S_{\mathrm{A}}\times\nabla\Theta \\ &= P_z\,\beta^{\Theta} T_{\mathrm{b}}^{\Theta}\left(\nabla_p S_{\mathrm{A}}\times\nabla_p\Theta\right)\cdot\mathbf{k} \\ &= g^{-1}N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k} \\ &\approx g^{-1}N^2 T_{\mathrm{b}}^{\Theta}\left(\nabla_a P\times\nabla_a\Theta\right)\cdot\mathbf{k} \end{aligned} \tag{3.13.2}$$

where P_z is simply the vertical gradient of pressure ($\mathrm{Pa\ m^{-1}}$) and $\nabla_n\Theta$ and $\nabla_p\Theta$ are the two-dimensional gradients of Θ in the neural tangent plane and in the horizontal plane (actually the isobaric surface) respectively. The gradients $\nabla_a P$ and $\nabla_a\Theta$ are taken in an approximately neutral surface.

Since $\alpha^{\theta}\nabla\theta - \beta^{\theta}\nabla S_{\mathrm{A}}$ and $\alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\nabla S_{\mathrm{A}}$ are exactly equal, neutral helicity can be defined in Eqn. (3.13.1) as the scalar product of this vector with its curl based on either formulation, so that (from the third line of Eqn. (3.13.2), and bearing in mind that $\nabla_n\Theta$ and $\nabla_n\theta$ are parallel vectors) we see that $T_{\mathrm{b}}^{\theta}\nabla_n\theta = T_{\mathrm{b}}^{\Theta}\nabla_n\Theta$, a result that we use in section 3.8 and in appendix A.14. Neutral helicity has units of $\mathrm{m^{-3}}$.

Interestingly, for given magnitudes of the epineutral gradients of pressure and Conservative Temperature, neutral helicity is maximized when these gradients are perpendicular since neutral helicity is proportional to $T_{\mathrm{b}}^{\Theta}\left(\nabla_n P\times\nabla_n\Theta\right)\cdot\mathbf{k}$ (see Eqn. (3.13.2)), while the dianeutral advection of thermobaricity, $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n P$, is maximized when $\nabla_n\Theta$ and $\nabla_n P$ are parallel (see section 3.8).

Because of the non-zero neutral helicity in the ocean, lateral motion following neutral tangent planes has the character of helical motion. That is, if we ignore the effects of diapycnal mixing processes (as well as ignoring cabbeling and thermobaricity), the mean flow around ocean gyres still passes through any well-defined "density" surface because of the helical nature of neutral trajectories, caused in turn by the non-zero neutral helicity. This dia-surface flow is expressed in Eqns. (A.25.4) and (A.25.6) in terms of the appropriate mean horizontal velocity and the difference between the slope of the neutral tangent plane and the slope of a well-defined "density" surface.

Neutral helicity in the world ocean is observed to be small in some sense. One way of visualizing this smallness of H^{n} is to examine all the hydrographic data in $\left(S_{\mathrm{A}},\Theta, p\right)$ space. When this is done for an entire ocean basin (for example, the whole of the combined North and South Atlantic oceans), and the data is spun in this three-dimensional $\left(S_{\mathrm{A}},\Theta, p\right)$ space, it is clear that the ocean hydrography lies close to a single surface in this $\left(S_{\mathrm{A}},\Theta, p\right)$ space. We will now show that if all the $\left(S_{\mathrm{A}},\Theta, p\right)$ data from the ocean lie exactly on a single surface $f\left(S_{\mathrm{A}},\Theta, p\right) = 0$ in $\left(S_{\mathrm{A}},\Theta, p\right)$ space, then this requires $\nabla S_{\mathrm{A}} \times \nabla\Theta \cdot \nabla P = 0$ everywhere in physical $\left(x, y, z\right)$ space. That is, we will prove that the "skinny" nature of the ocean hydrography in $\left(S_{\mathrm{A}},\Theta, p\right)$ space is a direct indication of the smallness of neutral helicity H^{n}.

Taking the spatial gradient of $f\left(S_{\mathrm{A}},\Theta, p\right) = 0$ in physical $\left(x, y, z\right)$ space we have $\nabla f = 0$ since f is zero at every point in physical $\left(x, y, z\right)$ space. Expanding ∇f in terms of the spatial gradients ∇S_{A}, $\nabla\Theta$, and ∇P, and taking the scalar product with $\nabla S_{\mathrm{A}} \times \nabla\Theta$ gives

$$f_P\big|_{S_{\mathrm{A}},\Theta} \nabla S_{\mathrm{A}} \times \nabla\Theta \cdot \nabla P = 0. \tag{3.13.3}$$

In the general case of $f_P \neq 0$, the result $\nabla S_{\mathrm{A}} \times \nabla\Theta \cdot \nabla P = 0$ is proven. In the special case $f_P = 0$, f is independent of P so that there is a simpler equation for the surface f, being $f\left(S_{\mathrm{A}},\Theta\right) = 0$, which is the equation for a single line on the $\left(S_{\mathrm{A}},\Theta\right)$ diagram; a single "water-mass" for the whole world ocean. In this case, changes in S_{A} are locally proportional to those of Θ so that $\nabla S_{\mathrm{A}} \times \nabla\Theta = \mathbf{0}$ which guarantees $\nabla S_{\mathrm{A}} \times \nabla\Theta \cdot \nabla P = 0$. Hence we have proven that the "skinniness" of the ocean hydrography in $\left(S_{\mathrm{A}},\Theta, p\right)$ space is a direct indication of the smallness of neutral helicity H^{n}.

The "skinny" nature of the North and South Atlantic hydrography is illustrated in Figure 6, which shows all the hydrographic data on the $S_{\mathrm{A}} - \Theta$ diagram at a pressure of $500\ \mathrm{dbar}$. This cut at constant pressure through the hydrographic data in three-dimensional $\left(S_{\mathrm{A}},\Theta, p\right)$ space, and similar cuts at different fixed pressures, show that the data from the whole physical $\left(x, y, z\right)$ volume of the North and South Atlantic lie close to a single surface in the three-dimensional $\left(S_{\mathrm{A}},\Theta, p\right)$ space. Figure 6 also illustrates the method of formation of one of Reid and Lynn's (1971) "isopycnals" and how the potential density anomaly with respect to the sea surface, σ_Θ, of $27.3\ \mathrm{kg\ m^{-3}}$ is matched to σ_1 of $31.938\ \mathrm{kg\ m^{-3}}$ in the Southern Ocean but to a different σ_Θ of $27.44\ \mathrm{kg\ m^{-3}}$ in the North Atlantic.

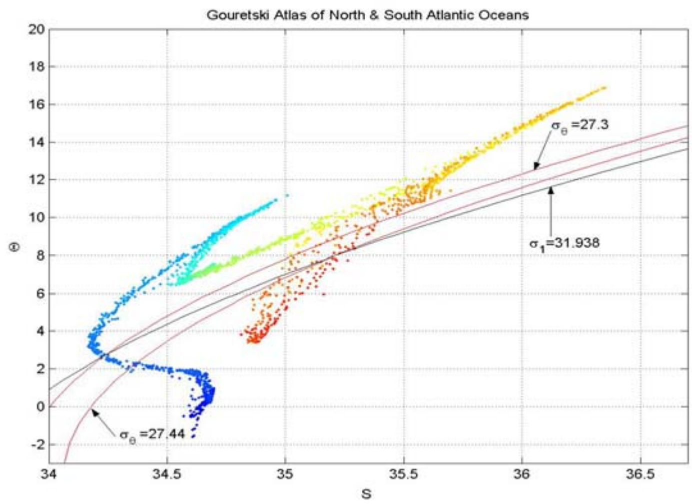

Figure 6. Hydrographic data from the ocean atlas of Gouretski and Koltermann (2004) for the North and South Atlantic at a pressure of 500 dbar. The colour of the data points indicates the latitude, from blue in the south through green at the equator to red in the north.

Neutral helicity is proportional to the component of the vertical shear of the geostrophic velocity ($\mathbf{v}_z$, the "thermal wind") in the direction of the temperature gradient along the neutral tangent plane $\nabla_n \Theta$, since, from Eqn. (3.12.3) and the third line of (3.13.2) we find that

$$H^{\mathrm{n}} = \rho T_{\mathrm{b}}^{\Theta} f \mathbf{v}_z \cdot \nabla_n \Theta . \tag{3.13.4}$$

In the evolution equation of potential vorticity defined with respect to potential density ρ^{θ} there is the baroclinic production term $\rho^{-2} \nabla \rho^{\theta} \cdot \nabla \rho \times \nabla p$ (Straub (1999)) and the first term in a Taylor series expansion for this baroclinic production term is proportional to neutral helicity and is given by (McDougall and Jackett (2007))

$$\rho^{-2} \nabla \rho^{\theta} \cdot \nabla \rho \times \nabla P \approx \left(P_{\mathrm{r}} - P \right) H^{\mathrm{n}} \tag{3.13.5}$$

where P_{r} is the reference pressure of the potential density. Similarly, the curl in a potential density surface of the horizontal pressure gradient term in the horizontal momentum equation, $\nabla_{\sigma} \times \left(\tfrac{1}{\rho} \nabla_z p \right)$, is given by (McDougall and Klocker (2010))

$$\nabla_{\sigma} \times \left(\tfrac{1}{\rho} \nabla_z P \right) \cdot \mathbf{k} = H^{\mathrm{n}} \left(P_{\mathrm{r}} - P \right) \left(-\frac{\partial \rho^{\Theta}}{\partial z} \right)^{-1} . \tag{3.13.6}$$

The fact that this curl is nonzero proves that a geostrophic streamfunction does not exist in a potential density surface.

The absolute velocity vector in the ocean can be written as a closed expression involving neutral helicity, and this expression is derived as follows. First the Eulerian-mean horizontal velocity is related directly to mixing processes by invoking the water-mass transformation equation (A.23.1), so that

$$\begin{aligned} \bar{\mathbf{v}} \cdot \nabla_n \hat{\Theta} \;=\; & \gamma_z \nabla_n \cdot \left(\gamma_z^{-1} K \nabla_n \hat{\Theta} \right) + K g N^{-2} \hat{\Theta}_z \left(C_{\mathrm{b}}^{\Theta} \nabla_n \hat{\Theta} \cdot \nabla_n \hat{\Theta} + T_{\mathrm{b}}^{\Theta} \nabla_n \hat{\Theta} \cdot \nabla_n P \right) \\ & + D \beta^{\Theta} g N^{-2} \hat{\Theta}_z^3 \frac{d^2 \hat{S}_{\mathrm{A}}}{d \hat{\Theta}^2} - \mathbf{\Psi}_z \cdot \nabla_n \hat{\Theta} - \hat{\Theta}_t \Big|_n , \end{aligned} \tag{3.13.7}$$

where the thickness-weighted mean velocity of density-coordinate averaging, $\hat{\mathbf{v}}$, has been written as $\hat{\mathbf{v}} = \bar{\mathbf{v}} + \mathbf{\Psi}_z$, that is, as the sum of the Eulerian-mean horizontal velocity $\bar{\mathbf{v}}$ and the quasi-Stokes eddy-induced horizontal velocity $\mathbf{\Psi}_z$ (McDougall and McIntosh (2001)). The quasi-Stokes vector streamfunction $\mathbf{\Psi}$ is usually expressed in terms of an imposed lateral diffusivity and the slope of the locally-referenced potential density surface (Gent *et al.*, (1995)). More generally, at least in a steady state when $\hat{\Theta}_t \big|_n$ is zero, the right-hand side of Eqn. (3.13.7) is due only to mixing processes and once the form of the lateral and vertical diffusivities are known, these terms are known in terms of the ocean's hydrography. Eqn. (3.13.9) is written more compactly as

$$\bar{\mathbf{v}} \cdot \boldsymbol{\tau} \;=\; v^{\perp} \qquad \text{where} \qquad \boldsymbol{\tau} \;\equiv\; \nabla_n \hat{\Theta} \Big/ \left|\nabla_n \hat{\Theta}\right| , \tag{3.13.8}$$

and $v^{\perp}$ is interpreted as being due to mixing processes.

The mean horizontal velocity $\bar{\mathbf{v}}$ is now split into the components along and across the contours of $\hat{\Theta}$ on the neutral tangent plane so that

$$\bar{\mathbf{v}} \;=\; v^{\parallel}\, \boldsymbol{\tau}\times\mathbf{k} \;+\; v^{\perp}\boldsymbol{\tau} \;, \tag{3.13.9}$$

where $v^{\parallel} \;=\; \bar{\mathbf{v}} \cdot \boldsymbol{\tau}\times\mathbf{k}$. Note that if $\boldsymbol{\tau}$ points northwards then $\boldsymbol{\tau}\times\mathbf{k}$ points eastward. The expression $\bar{\mathbf{v}} \cdot \boldsymbol{\tau} \;=\; v^{\perp}$ of Eqn. (3.13.8) is now vertically differentiated to obtain

$$\bar{\mathbf{v}} \cdot \boldsymbol{\tau}_z \;=\; -\bar{\mathbf{v}}_z \cdot \boldsymbol{\tau} \;+\; v_z^{\perp} \;\;=\; -\tfrac{N^2}{fg\rho}\, \mathbf{k}\times\nabla_n P \cdot \boldsymbol{\tau} \;+\; v_z^{\perp} , \tag{3.13.10}$$

where we have used the "thermal wind" equation (3.12.3), $\bar{\mathbf{v}}_z \;=\; \tfrac{N^2}{fg\rho}\, \mathbf{k}\times\nabla_n P$. We will now show that the left-hand side of this equation is $-\,\phi_z v^{\parallel}$ where ϕ_z is the rate of rotation of the direction of the unit vector $\boldsymbol{\tau}$ with respect to height (in radians per metre). By expressing the two-dimensional unit vector $\boldsymbol{\tau}$ in terms of the angle ϕ (measured counter-clockwise) of $\boldsymbol{\tau}$ with respect to due north so that $\boldsymbol{\tau} = \left(-\sin\phi, \cos\phi\right)$, we see that $\boldsymbol{\tau}\times\mathbf{k} \;=\; \left(\cos\phi, \sin\phi\right)$, $\boldsymbol{\tau}_z \;=\; -\,\phi_z\, \boldsymbol{\tau}\times\mathbf{k}$ and $\mathbf{k}\cdot\boldsymbol{\tau}\times\boldsymbol{\tau}_z \;=\; \phi_z$. Interestingly, ϕ_z is also equal to minus the helicity of $\boldsymbol{\tau}$ (and to minus the helicity of $\boldsymbol{\tau}\times\mathbf{k}$), that is, $\phi_z \;=\; -\,\boldsymbol{\tau}\cdot\nabla\times\boldsymbol{\tau} \;=\; -\left(\boldsymbol{\tau}\times\mathbf{k}\right)\cdot\nabla\times\left(\boldsymbol{\tau}\times\mathbf{k}\right)$, where the helicity of a vector is defined to be the scalar product of the vector with its curl. From the velocity decomposition (3.13.9) and the equation $\boldsymbol{\tau}_z \;=\; -\,\phi_z\, \boldsymbol{\tau}\times\mathbf{k}$ we see that the left-hand side of Eqn. (3.13.10), $\bar{\mathbf{v}}\cdot\boldsymbol{\tau}_z$, is $-\,\phi_z v^{\parallel}$, hence $v^{\parallel}$ can be expressed as

$$v^{\parallel} \;=\; \frac{N^2}{fg\rho}\,\frac{\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}}{\phi_z} \;-\; \frac{v_z^{\perp}}{\phi_z} \qquad \text{or} \qquad v^{\parallel} \;=\; \frac{H^{\mathrm{n}}}{\phi_z \rho f\, T_{\mathrm{b}}^{\Theta}\left|\nabla_n\hat{\Theta}\right|} \;-\; \frac{v_z^{\perp}}{\phi_z} , \tag{3.13.11}$$

where we have used the definition of neutral helicity H^{n}, Eqn. (3.13.2). The expression for both horizontal components of the Eulerian-mean horizontal velocity vector $\bar{\mathbf{v}}$ is

$$\boxed{\bar{\mathbf{v}} \;=\; \left\{ \frac{N^2}{fg\rho}\,\frac{\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}}{\phi_z} \;-\; \frac{v_z^{\perp}}{\phi_z} \right\} \boldsymbol{\tau}\times\mathbf{k} \;+\; v^{\perp}\boldsymbol{\tau} ,} \tag{3.13.12}$$

and the horizontal velocity due to solely the two mixing terms can be expressed as

$$-\frac{v_z^{\perp}}{\phi_z}\,\boldsymbol{\tau}\times\mathbf{k} \;+\; v^{\perp}\boldsymbol{\tau} \;=\; \frac{(v^{\perp})^2}{\phi_z}\left(\frac{\boldsymbol{\tau}\times\mathbf{k}}{v^{\perp}}\right)_z , \text{ which has the magnitude } \left|\frac{1}{\phi_z}\left(v^{\perp}\boldsymbol{\tau}\times\mathbf{k}\right)_z\right| . \tag{3.13.13}$$

Equation (3.13.12) for the Eulerian-mean horizontal velocity $\bar{\mathbf{v}}$ shows that in the absence of mixing processes (so that $v^{\perp} = v_z^{\perp} = 0$) and so long as (i) the epineutral $\hat{\Theta}$ contours do spiral in the vertical and (ii) $\left|\nabla_n\hat{\Theta}\right|$ is not zero, then neutral helicity H^{n} (which is proportional to $\mathbf{k}\cdot\nabla_n P\times\boldsymbol{\tau}$) is required to be non-zero in the ocean whenever the ocean is not motionless. Neutral helicity arises in this context because it is proportional to the component of the thermal wind vector $\bar{\mathbf{v}}_z$ in the direction across the $\hat{\Theta}$ contour on the neutral tangent plane (see Eqn. (3.13.4)). This derivation of the expression for the mean absolute horizontal velocity vector $\bar{\mathbf{v}}$ is based on McDougall (1995) and Zika *et al*. (2010a).

A.14 Advective and diffusive "heat" fluxes

In section 3.23 and appendices A.8 and A.13 the First Law of Thermodynamics is shown to be practically equivalent to the conservation equation (A.21.15) for Conservative Temperature Θ. We have emphasized that this means that the advection of "heat" is very accurately given as the advection of $c_p^0\Theta$. In this way $c_p^0\Theta$ can be regarded as the "heat content" per unit mass of seawater and the error involved with making this association is approximately 1% of the error in assuming that either $c_p^0\theta$ or $c_p\left(S_\mathrm{A},\theta,0\,\mathrm{dbar}\right)\theta$ is the "heat content" per unit mass of seawater (see also appendix A.21 for a discussion of this point).

The turbulent flux of a "potential" property can be thought of as the exchange of parcels of equal mass but contrasting values of the "potential" property, and the turbulent flux can be parameterized as being down the gradient of the "potential" property. The conservative form of Eqn. (A.21.15) implies that the turbulent flux of heat should be directed down the mean gradient of Conservative Temperature rather than down the mean gradient of potential temperature. In this appendix we quantify the ratio of the mean gradients of potential temperature and Conservative Temperature.

Consider first the respective temperature gradients along the neutral tangent plane. From Eqn. (3.11.2) we find that

$$\left(\alpha^{\theta}/\beta^{\theta}\right)\nabla_n\theta = \nabla_n S_\mathrm{A} = \left(\alpha^{\Theta}/\beta^{\Theta}\right)\nabla_n\Theta, \tag{A.14.1}$$

so that the epineutral gradients of θ and Θ are related by the ratios of their respective thermal expansion and saline contraction coefficients, namely

$$\nabla_n\theta = \frac{\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\left(\alpha^{\theta}/\beta^{\theta}\right)}\nabla_n\Theta. \tag{A.14.2}$$

This proportionality factor between the parallel two-dimensional vectors $\nabla_n\theta$ and $\nabla_n\Theta$ is readily calculated and illustrated graphically. Before doing so we note two other equivalent expressions for this proportionality factor.

The epineutral gradients of θ, Θ and S_A are related by (using $\theta = \hat{\theta}\left(S_\mathrm{A},\Theta\right)$)

$$\nabla_n\theta = \hat{\theta}_{\Theta}\nabla_n\Theta + \hat{\theta}_{S_\mathrm{A}}\nabla_n S_\mathrm{A}, \tag{A.14.3}$$

and using the neutral relationship $\nabla_n S_\mathrm{A} = \left(\alpha^{\Theta}/\beta^{\Theta}\right)\nabla_n\Theta$ we find

$$\nabla_n\theta = \left(\hat{\theta}_{\Theta} + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_\mathrm{A}}\right)\nabla_n\Theta. \tag{A.14.4}$$

Also, in section 3.13 we found that $T_b^{\theta}\nabla_n\theta = T_b^{\Theta}\nabla_n\Theta$, so that we find the expressions

$$\frac{\left|\nabla_n\theta\right|}{\left|\nabla_n\Theta\right|} = \frac{\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\left(\alpha^{\theta}/\beta^{\theta}\right)} = \frac{T_b^{\Theta}}{T_b^{\theta}} = \hat{\theta}_{\Theta} + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_\mathrm{A}}, \tag{A.14.5}$$

and it can be shown that $\alpha^{\Theta}/\alpha^{\theta} = \hat{\theta}_{\Theta}$ and $\beta^{\theta}/\beta^{\Theta} = \left(1 + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_\mathrm{A}}/\hat{\theta}_{\Theta}\right)$, that is, $\beta^{\theta} = \beta^{\Theta} + \alpha^{\Theta}\,\hat{\theta}_{S_\mathrm{A}}/\hat{\theta}_{\Theta}$. The ratios $\alpha^{\Theta}/\alpha^{\theta}$ and $\beta^{\theta}/\beta^{\Theta}$ have been plotted by Graham and McDougall (2013); interestingly $\alpha^{\Theta}/\alpha^{\theta}$ is approximately a linear function of S_A while $\beta^{\theta}/\beta^{\Theta}$ is approximately a function of only Θ. The partial derivatives $\hat{\theta}_{\Theta}$ and $\hat{\theta}_{S_\mathrm{A}}$ in the last part of Eqn. (A.14.5) are both independent of pressure while $\alpha^{\Theta}/\beta^{\Theta}$ is a function of pressure. The ratio, Eqn. (A.14.5), of the epineutral gradients of θ and Θ is shown in Figure A.14.1 at $p = 0$, indicating that the epineutral gradient of potential temperature is sometimes more that 1% different to that of Conservative Temperature. This ratio $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ is only a weak function of pressure. This ratio, $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ (i.e. Eqn. (A.14.5)), is available in the GSW Oceanographic Toolbox as function **gsw_ntp_pt_vs_CT_ratio**.

Similarly to Eqn. (A.14.3), the vertical gradients are related by

$$\theta_z = \hat{\theta}_{\Theta}\Theta_z + \hat{\theta}_{S_\mathrm{A}}S_{\mathrm{A}_z}, \tag{A.14.6}$$

and using the definition, Eqn. (3.15.1), of the stability ratio we find that

$$\frac{\theta_z}{\Theta_z} = \hat{\theta}_{\Theta} + R_{\rho}^{-1}\left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_\mathrm{A}}. \tag{A.14.7}$$

For values of the stability ratio R_{ρ} close to unity, the ratio θ_z/Θ_z is close to the values of $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ shown in Figure A.14.1.

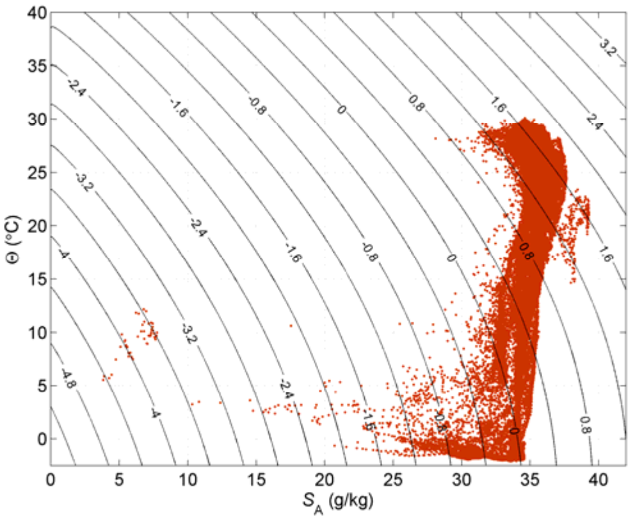

Figure A.14.1. Contours of $\left(\left|\nabla_n\theta\right| / \left|\nabla_n\Theta\right| \; - \; 1\right) \times 100\%$ at $p = 0$, showing the percentage difference between the epineutral gradients of θ and Θ. The red dots are from the ocean atlas of Gouretski and Koltermann (2004) at $p = 0$.

As noted in section 3.8 the dianeutral advection of thermobaricity is the same when quantified in terms of θ as when done in terms of Θ. The same is not true of the dianeutral velocity caused by cabbeling. The ratio of the cabbeling dianeutral velocity calculated using potential temperature to that using Conservative Temperature is given by $\left(C_{\mathrm{b}}^{\theta}\nabla_n\theta\cdot\nabla_n\theta\right) / \left(C_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n\Theta\right)$ (see section 3.9) which can be expressed as

$$\frac{C_{\mathrm{b}}^{\theta}\left|\nabla_n\theta\right|^2}{C_{\mathrm{b}}^{\Theta}\left|\nabla_n\Theta\right|^2} = \frac{C_{\mathrm{b}}^{\theta}}{C_{\mathrm{b}}^{\Theta}}\frac{\left(\alpha^{\Theta}/\beta^{\Theta}\right)^2}{\left(\alpha^{\theta}/\beta^{\theta}\right)^2} = \frac{C_{\mathrm{b}}^{\theta}}{C_{\mathrm{b}}^{\Theta}}\left(\frac{T_{\mathrm{b}}^{\Theta}}{T_{\mathrm{b}}^{\theta}}\right)^2 = \frac{C_{\mathrm{b}}^{\theta}}{C_{\mathrm{b}}^{\Theta}}\left(\hat{\theta}_{\Theta} + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_{\mathrm{A}}}\right)^2, \qquad \text{(A.14.8)}$$

and this is contoured in Fig. A.14.2. While the ratio of Eqn. (A.14.8) is not exactly unity, it varies relatively little in the oceanographic range, indicating that the dianeutral advection due to cabbeling estimated using θ or Θ are within half a percent of each other at $p = 0$.

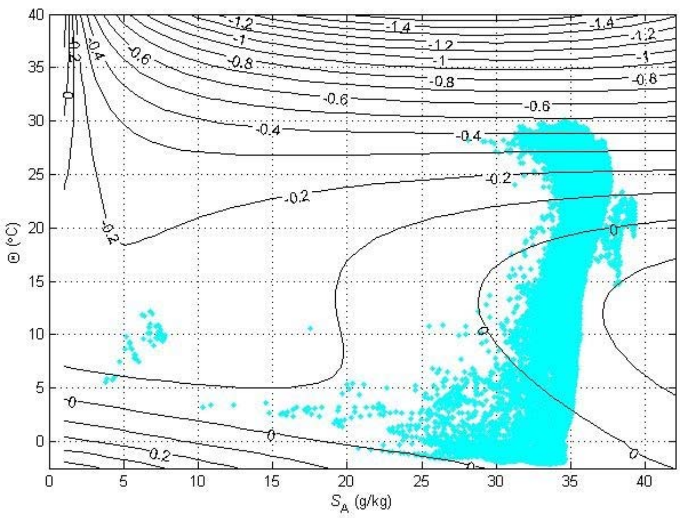

Figure A.14.2. Contours of the percentage difference of $\left(C_{\mathrm{b}}^{\theta}\left|\nabla_n\theta\right|^2\right) / \left(C_{\mathrm{b}}^{\Theta}\left|\nabla_n\Theta\right|^2\right)$ from unity at $p = 0$ dbar.