

## Notes on the function gsw\_pot\_enthalpy\_ice\_freezing\_first\_derivatives(SA,p)

This function, **gsw\_pot\_enthalpy\_ice\_freezing\_first\_derivatives(SA,p)**, evaluates the first derivatives of the potential enthalpy (reference pressure of 0 dbar) of ice Ih at the temperature at which it melts into seawater of Absolute Salinity  $S_A$  at pressure  $p$ .

The specific enthalpy of ice  $h^{\text{lh}}$  is given in terms of the Gibbs function of ice whose independent variables are *in situ* temperature  $t$  and pressure  $p$ , so that the native functional form of specific enthalpy is  $h^{\text{lh}} = h^{\text{lh}}(t, p)$ . The potential enthalpy of ice,  $\hat{h}^{\text{lh}}$ , and the potential temperature of ice will always be considered with respect to the reference pressure of 0 dbar, so that  $\theta^{\text{lh}} = \theta^{\text{lh}}(t, p)$  and

$$\theta^{\text{lh}} = \theta^{\text{lh}}(t, p) \quad \text{and} \quad \hat{h}^{\text{lh}} = h^{\text{lh}}(\theta^{\text{lh}}, 0) = \hat{h}^{\text{lh}}(\theta^{\text{lh}}). \quad (1a,b)$$

The *in situ* temperature at which seawater is in thermodynamic equilibrium with ice (the freezing temperature) can be expressed as a function of the Absolute Salinity  $S_A$  of the interstitial seawater and the ambient pressure  $p$  as

$$t_{\text{freezing}} = t_{\text{freezing}}(S_A, p). \quad (2)$$

The potential enthalpy of ice at the freezing temperature is then given by combining the above functional forms as

$$\hat{h}_{\text{freezing}}^{\text{lh}} = h^{\text{lh}}(\theta_{\text{freezing}}^{\text{lh}}, 0) = h^{\text{lh}}(\theta^{\text{lh}}[t_{\text{freezing}}(S_A, p), p], 0). \quad (3)$$

From this equation we can derive expressions for the desired partial derivatives of the potential enthalpy of ice with respect to (i) Absolute Salinity and (ii) pressure as

$$\left. \frac{\partial \hat{h}_{\text{freezing}}^{\text{lh}}}{\partial S_A} \right|_p = \left. \frac{dh^{\text{lh}}}{d\theta^{\text{lh}}} \frac{\partial \theta^{\text{lh}}}{\partial T} \right|_p \left. \frac{\partial t_{\text{freezing}}}{\partial S_A} \right|_p, \quad (4)$$

and

$$\left. \frac{\partial \hat{h}_{\text{freezing}}^{\text{lh}}}{\partial P} \right|_{S_A} = \left. \frac{dh^{\text{lh}}}{d\theta^{\text{lh}}} \left( \left. \frac{\partial \theta^{\text{lh}}}{\partial T} \right|_p \left. \frac{\partial t_{\text{freezing}}}{\partial P} \right|_{S_A} + \left. \frac{\partial \theta^{\text{lh}}}{\partial P} \right|_T \right). \quad (5)$$

We now need to evaluate the right-hand sides of these two equations. Expressions for the two partial derivatives of the freezing *in situ* temperature are well-known in the TEOS-10 literature, and here we repeat these expressions from Eqns. (C3a) and (C3b) of McDougall et al. (2014), (here the Gibbs function of ice is  $g^{\text{lh}} = g^{\text{lh}}(t, p)$  and the Gibbs function of seawater is  $g = g(S_A, t, p)$ )

$$\left. \frac{\partial t_{\text{freezing}}}{\partial S_A} \right|_p = \frac{S_A g_{S_A S_A}}{g_T - S_A g_{T S_A} - g_T^{\text{lh}}}, \quad (6)$$

and

$$\left. \frac{\partial t_{\text{freezing}}}{\partial P} \right|_{S_A} = - \frac{g_P - S_A g_{P S_A} - g_P^{\text{lh}}}{g_T - S_A g_{T S_A} - g_T^{\text{lh}}}. \quad (7)$$

These partial derivatives are available in the GSW Oceanographic Toolbox from the function **gsw\_t\_freezing\_first\_derivatives(SA,p,saturation\_fraction)**, and in this work on frazil ice and seawater mixtures we will always take the saturation\_fraction of air in the seawater to be zero in order to obtain thermodynamic consistency between the Gibbs functions.

The first term on the right-hand sides of both Eqns. (4) and (5) can be readily calculated from the Gibbs function of ice Ih as the isobaric specific heat capacity evaluated at a temperature of  $\theta^{\text{lh}}$  and at 0 dbar, namely

$$\frac{d\hat{h}^{\text{lh}}}{d\theta^{\text{lh}}} = c_p^{\text{lh}}(\theta^{\text{lh}}, 0), \quad (8)$$

and this derivative can be evaluated using the GSW function `gsw_cp_ice(t,p)`.

In order to evaluate Eqns. (4) and (5) we also need

$$\left. \frac{\partial \theta^{\text{lh}}}{\partial T} \right|_p \quad \text{and} \quad \left. \frac{\partial \theta^{\text{lh}}}{\partial P} \right|_T, \quad (9)$$

and we find these by differentiating the entropy equality

$$\eta^{\text{lh}}(t, p) = \eta^{\text{lh}}(\theta^{\text{lh}}[t, p, 0], 0), \quad (10)$$

first with respect to *in situ* temperature and then with respect to pressure (see appendix A.15 of IOC et al. (2010) for motivation based on the case of seawater rather than of ice Ih). This procedure leads to

$$\left. \frac{\partial \theta^{\text{lh}}}{\partial T} \right|_p = \frac{g_{TT}^{\text{lh}}(t, p)}{g_{TT}^{\text{lh}}(\theta^{\text{lh}}, 0)} = \frac{(T_0 + \theta^{\text{lh}}) c_p^{\text{lh}}(t, p)}{(T_0 + t) c_p^{\text{lh}}(\theta, 0)}, \quad (11)$$

and

$$\left. \frac{\partial \theta^{\text{lh}}}{\partial P} \right|_T = \frac{g_{TP}^{\text{lh}}(t, p)}{g_{TT}^{\text{lh}}(\theta^{\text{lh}}, 0)}. \quad (12)$$

We are done. Now we can combine the above expressions, and achieve some benefit of cancellation, so that the desired expressions of Eqns. (4) and (5) become

$$\boxed{\left. \frac{\partial \hat{h}_{\text{freezing}}^{\text{lh}}}{\partial S_A} \right|_p = \frac{(T_0 + \theta^{\text{lh}})}{(T_0 + t)} c_p^{\text{lh}}(t, p) \left. \frac{\partial t_{\text{freezing}}}{\partial S_A} \right|_p} \quad (13)$$

and

$$\boxed{\left. \frac{\partial \hat{h}_{\text{freezing}}^{\text{lh}}}{\partial P} \right|_{S_A} = \frac{(T_0 + \theta^{\text{lh}})}{(T_0 + t)} c_p^{\text{lh}}(t, p) \left. \frac{\partial t_{\text{freezing}}}{\partial P} \right|_{S_A} - (T_0 + \theta^{\text{lh}}) g_{TP}^{\text{lh}}(t, p)}. \quad (14)$$

These Eqns. (13) and (14) are evaluated in the present GSW code,

`gsw_pot_enthalpy_ice_freezing_first_derivatives(SA,p)`, with every occurrence of *in situ* temperature  $t$  being the freezing the *in situ* temperature  $t_{\text{freezing}} = t_{\text{freezing}}(S_A, p)$  and every occurrence of the potential temperature of ice  $\theta^{\text{lh}}$  being  $\theta_{\text{freezing}}^{\text{lh}} = \theta_{\text{freezing}}^{\text{lh}}(S_A, p)$ .

## References

- IAPWS, 2008: Release on the IAPWS Formulation 2008 for the Thermodynamic Properties of Seawater. The International Association for the Properties of Water and Steam. Berlin, Germany, September 2008, available from [www.iapws.org](http://www.iapws.org). This Release is referred to in the text as **IAPWS-08**.
- IAPWS, 2009: Supplementary Release on a Computationally Efficient Thermodynamic Formulation for Liquid Water for Oceanographic Use. The International Association for the Properties of Water and Steam. Doorwerth, The Netherlands, September 2009, available from <http://www.iapws.org>. This Release is referred to in the text as **IAPWS-09**.
- IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from <http://www.TEOS-10.org>

McDougall, T. J., P. M. Barker, R. Feistel and B. K. Galton-Fenzi, 2014: Melting of ice and sea ice into seawater, and frazil ice formation. *Journal of Physical Oceanography*, **44**, 1751-1775.