

Notes on the GSW code

gsw_z_from_p

for calculating height z from pressure p

Height z is measured positive upwards, so it is negative in the ocean. First, note that we use the following version of specific volume anomaly,

$$\delta = \hat{v}(S_A, \Theta, p) - \hat{v}(S_{SO}, 0^\circ\text{C}, p). \quad (1)$$

That is, the reference Absolute Salinity is the Absolute Salinity of the Standard Ocean, $S_{SO} \equiv 35.165\,04 \text{ g kg}^{-1}$, and the reference “temperature” is a fixed value of Conservative Temperature of zero degrees Celsius. Dynamic height anomaly Ψ is then defined by Eqn. (3.27.1) of IOC *et al.* (2010) as follows

$$\Psi = - \int_{P_0}^P \delta(p') dp', \quad (2)$$

where $P_0 = 101\,325 \text{ Pa}$ is the standard atmosphere pressure.

The vertical integral of the hydrostatic equation ($P_z = -g\rho$ or $g = -vP_z$) is (from Eqn. (3.32.3) of the TEOS-10 Manual (IOC *et al.* (2010)))

$$\begin{aligned} \int_0^z g(z') dz' &= \Phi^0 - \int_{P_0}^P v(p') dp' = - \int_{P_0}^P \hat{v}(S_{SO}, 0^\circ\text{C}, p') dp' + \Psi + \Phi^0 \\ &= -\hat{h}(S_{SO}, 0^\circ\text{C}, p) + \Psi + \Phi^0, \end{aligned} \quad (3.23.3)$$

Here Φ^0 is the geopotential at zero sea pressure on this vertical cast. We use the 76-term based expression for enthalpy (Eqn. (A.30.6) of the TEOS-10 Manual), recognizing that because $\Theta = 0^\circ\text{C}$ many of the coefficients are zero, so the evaluation of Eqn. (A.30.6) is less computationally expensive than it may appear. The library function **gsw_enthalpy_SSO_0(p)** is used to evaluate $\hat{h}^{76}(S_{SO}, 0^\circ\text{C}, p)$ efficiently at these fixed values of Absolute Salinity and Conservative Temperature.

Writing the gravitational acceleration of Eqn. (D.3) of IOC *et al.* (2010) as

$$g = g(\phi, z) = g(\phi, 0)(1 - \gamma z), \quad (4)$$

we see that Eqn. (3.32.3) becomes

$$\hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p) - \Psi - \Phi^0 + g(\phi, 0)\left(z - \frac{1}{2}\gamma z^2\right) = 0. \quad (5)$$

When the **gsw_z_from_p** code is called with two arguments, as in **gsw_z_from_p(p,lat)**, $\Psi + \Phi^0$ is ignored in Eqn. (5) and this quadratic expression is solved for the height z . We do this using the standard quadratic solution equation, but for z^{-1} . This is done so that the result is accurate as pressure tends to zero, and so that the answer also converges to the correct solution when the quadratic term γ tends to zero (since there may be some applications where it is preferable to assume that the gravitational acceleration is depth-independent). Hence we evaluate z from the equation

$$z = - \frac{2\left(\hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p) - \Psi - \Phi^0\right)}{g(\phi, 0) + \sqrt{g^2(\phi, 0) + 2\gamma g(\phi, 0)\left(\hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p) - \Psi - \Phi^0\right)}}. \quad (6)$$

Note again that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to be dynamic height Ψ and the geopotential at

zero pressure Φ^0 is taken to be zero. When the code is called with four arguments the third argument is taken to be Ψ and the fourth Φ^0 . The dynamic height anomaly Ψ can be evaluated using the GSW function `gsw_geo_strf_dyn_height`, noting that the reference pressure in the call to this function must be zero sea pressure.

Note that in Eqn. (5) the last term, $g(\phi, 0) \left(z - \frac{1}{2} \gamma z^2 \right)$, can be written as $z \bar{g}$ where \bar{g} is the mean gravitational acceleration between $z = 0$ and the height concerned. Recognizing this, the height z output from this algorithm is also equal to

$$z = - \frac{\left(\hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p) - \Psi - \Phi^0 \right)}{\bar{g}}. \quad (7)$$

Notes on the GSW code

`gsw_p_from_z`

for calculating pressure p from height z

In the `gsw_p_from_z` code we evaluate pressure p using the modified Newton-Raphson" iteration procedure of McDougall and Wotherspoon (2014) so that the pressure so obtained is exactly consistent with the "forward" calculation of z from p via the function `gsw_z_from_p`.

When the `gsw_p_from_z` code is called with two arguments, as in `gsw_p_from_z(z, lat)`, we ignore $\Psi + \Phi^0$ while solving Eqn. (8) below. Note again that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to dynamic height Ψ and the geopotential at zero pressure Φ^0 is taken to be zero. When the code is called with four arguments the third argument is taken to be Ψ and the fourth Φ^0 . The dynamic height anomaly Ψ can be evaluated using the GSW function `gsw_geo_strf_dyn_height`, noting that the reference pressure in the call to this function must be zero sea pressure.

A good starting point for pressure is found by using the Saunders (1981) quadratic expression relating depth to a quadratic of pressure; we solve this quadratic using the standard quadratic solution formula but for p^{-1} instead of for p , so that the solution is well-behaved as z goes to zero.

Hence, given z , we have a zeroth estimate of pressure, p_0 , from the Saunders (1981) quadratic expression. Now we want to solve (see Eqn. (3.32.3) of the TEOS-10 Manual, IOC *et al.* (2010)),

$$f(p) = 0, \quad \text{where} \quad f(p) = \hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p) - \Psi - \Phi^0 + g(\phi, 0) \left(z - \frac{1}{2} \gamma z^2 \right). \quad (8)$$

The derivative of $f(p)$ is approximately

$$f'(p) = 10^4 \hat{v}^{75}(S_{SO}, 0^\circ\text{C}, p), \quad (9)$$

and this is available from the 75-term polynomial function expression for seawater specific volume (Roquert *et al.*, 2015) (and since $\Theta = 0^\circ\text{C}$, $\hat{v}^{75}(S_{SO}, 0^\circ\text{C}, p)$ is particularly simple to evaluate using the library function `gsw_specvol_SSO_0(p)`). The factor of 10^4

in Eqn. (9) is because we want to solve for pressure in dbar rather than in the natural SI unit for pressure of Pa. That is, Eqn. (9) is the derivative of $f(p)$ with respect to pressure p in dbar.

After finding p_0 we evaluate $f(p_0) = \hat{h}^{75}(S_{SO}, 0^\circ\text{C}, p_0) - \Psi - \Phi^0 + g(\phi, 0) \left(z - \frac{1}{2}\gamma z^2 \right)$, then calculate $f'(p_0) = 10^4 \hat{v}^{75}(S_{SO}, 0^\circ\text{C}, p_0)$ and use these values of $f(p_0)$ and $f'(p_0)$ to form an intermediate pressure estimate p_1 as (this is a standard Newton's method iteration)

$$p_1 = p_0 - f(p_0)/f'(p_0) . \quad (8)$$

Then we form $p_m = 0.5(p_0 + p_1)$ and evaluate $f'(p_m) = 10^4 \hat{v}^{75}(S_{SO}, 0^\circ\text{C}, p_m)$ and use $f(p_0)$ and $f'(p_m)$ to calculate p_2 from

$$p_2 = p_0 - f(p_0)/f'(p_m) . \quad (9)$$

This is one full step of the "modified Newton-Raphson" iteration procedure of McDougall and Wotherspoon (2014), and this one modified step gives pressure to better than 1.6×10^{-10} dbar (which is essentially machine precision) down to a height z of -8000m. The **gsw_p_from_z** function performs this one full iteration of the modified Newton-Raphson iteration.

References

- McDougall T. J. and S. J. Wotherspoon, 2014: A simple modification of Newton's method to achieve convergence of order $1+\sqrt{2}$. *Applied Mathematics Letters*, **29**, 20-25. <http://dx.doi.org/10.1016/j.aml.2013.10.008>
- Saunders, P. M, 1981: Practical conversion of pressure to depth. *Journal of Physical Oceanography*, **11**, 573-574.
- Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. *Ocean Modelling*, **90**, pp. 29-43. <http://dx.doi.org/10.1016/j.ocemod.2015.04.002>

Below is section 3.32 of the TEOS-10 Manual (IOC *et al.* (2010)).

3.32 Pressure to height conversion

The vertical integral of the hydrostatic equation ($g = -vP_z$) can be written as

$$\begin{aligned} \int_0^z g(z') dz' &= \Phi^0 - \int_{P_0}^P v(p') dP' = - \int_{P_0}^P \hat{v}(S_{SO}, 0^\circ\text{C}, p') dP' + \Psi + \Phi^0 \\ &= - \hat{h}(S_{SO}, 0^\circ\text{C}, p) + \Psi + \Phi^0, \end{aligned} \quad (3.32.1)$$

where the dynamic height anomaly Ψ is expressed in terms of the specific volume anomaly $\hat{\delta} = \hat{v}(S_A, \Theta, p) - \hat{v}(S_{SO}, 0^\circ\text{C}, p)$ by

$$\Psi = - \int_{P_0}^P \hat{\delta}(p') dP', \quad (3.32.2)$$

where $P_0 = 101\,325\text{Pa}$ is the standard atmosphere pressure. Writing the gravitational acceleration of Eqn. (D.3) as $g = g(\phi, z) = g(\phi, 0) (1 - \gamma z)$, the left-hand side of Eqn.

(3.32.1) becomes $g(\phi, 0) \left(z - \frac{1}{2} \gamma z^2 \right)$, and using the 75-term expression for the specific enthalpy of Standard Seawater, Eqn. (3.32.1) becomes

$$\hat{h}^{75}(S_{\text{SO}}, 0^\circ\text{C}, p) - \Psi - \Phi^0 + g(\phi, 0) \left(z - \frac{1}{2} \gamma z^2 \right) = 0. \quad (3.32.3)$$

This is the equation that is solved for height z in the GSW function `gsw_z_from_p`. It is traditional to ignore $\Psi + \Phi^0$ when converting between pressure and height, and this can be done by simply calling this function with only two arguments, as in `gsw_z_from_p(p, lat)`. Ignoring $\Psi + \Phi^0$ makes a difference to z of up to 4m at 5000 dbar. Note that height z is negative in the ocean. When the code is called with three arguments, the third argument is taken to be $\Psi + \Phi^0$ and this is used in the solution of Eqn. (3.32.3). Dynamic height anomaly Ψ can be evaluated using the GSW function `gsw_geo_strf_dyn_height`. The GSW function `gsw_p_from_z` is the exact inverse function of `gsw_z_from_p`; these functions yield outputs that are consistent with each other to machine precision.

When vertically integrating the hydrostatic equation $P_z = -g\rho$ in the context of an ocean model where Absolute Salinity S_A and Conservative Temperature Θ are piecewise constant in the vertical, the geopotential (Eqn. (3.24.2))

$$\Phi = \int_0^z g(z') dz' = \Phi^0 - \int_{P_0}^P v(p') dP', \quad (3.32.4)$$

can be evaluated as a series of exact differences. If there are a series of layers of index i separated by pressures p^i and p^{i+1} (with $p^{i+1} > p^i$) then the integral can be expressed (making use of (3.7.5), namely $h_p|_{S_A, \Theta} = \hat{h}_p = v$) as a sum over n layers of the differences in specific enthalpy so that

$$\Phi = \Phi^0 - \int_{P_0}^P v(p') dP' = \Phi^0 - \sum_{i=1}^n \left[\hat{h}(S_A^i, \Theta^i, p^{i+1}) - \hat{h}(S_A^i, \Theta^i, p^i) \right]. \quad (3.32.5)$$

The difference in enthalpy at two different pressures for given values of S_A and Θ is available in the GSW Oceanographic Toolbox via the function `gsw_enthalpy_diff`. The summation of a series of such differences in enthalpy occurs in the GSW functions to evaluate two geostrophic streamfunctions from piecewise-constant vertical property profiles, `gsw_geo_strf_dyn_height_pc` and `gsw_geo_strf_isopycnal_pc`.