

Notes on the function gsw_dynamic_enthalpy(SA,CT,p)

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Young (2010) has defined dynamic enthalpy h^\dagger to be the difference between enthalpy and potential enthalpy, that is, $h - h^0 = h - c_p^0 \Theta$. Hence dynamic enthalpy h^\dagger is also equal to the following pressure integral of specific volume for a seawater parcel which does not exchange heat or salt as its pressure is changed during the integration, (see Eqn. (3.2.1) of the TEOS-10 Manual, IOC *et al.* (2010))

$$\hat{h}^\dagger(S_A, \Theta, p) = h(S_A, \Theta, p) - c_p^0 \Theta = \int_{P_0}^P \hat{v}(S_A, \Theta, p') dP', \quad (1)$$

The lower limit of the integration is $P_0 \equiv 101\,325$ Pa and the pressure integral is done with pressure in Pa (not dbar). Enthalpy and dynamic enthalpy have units of J kg^{-1} .

The identical functions **gsw_dynamic_enthalpy** evaluates the dynamic enthalpy of seawater as a function of Absolute Salinity, Conservative Temperature and pressure using the 75-term expression, $\hat{v}(S_A, \Theta, p)$. This 75-term polynomial expression for specific volume is discussed in Roquet *et al.* (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al.* (2010)). For dynamical oceanography we may take the 75-term polynomial expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

References

- IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from <http://www.TEOS-10.org>
- McDougall, T.J., D.R. Jackett, D.G. Wright and R. Feistel, 2003: Accurate and computationally efficient algorithms for potential temperature and density of seawater. *J. Atmosph. Ocean. Tech.*, **20**, pp. 730-741.
- Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. *Ocean Modelling*, **90**, pp. 29-43. <http://dx.doi.org/10.1016/j.ocemod.2015.04.002>
- Young, W. R., 2010: Dynamic enthalpy, Conservative Temperature, and the seawater Boussinesq approximation. *Journal of Physical Oceanography*, **40**, 394–400.

Here follows sections 3.2 and 3.3, and appendices A.30 and K of the TEOS-10 Manual (IOC *et al.* (2010)).

3.2 Potential enthalpy

Potential enthalpy h^0 is the enthalpy that a fluid parcel would have if its pressure were changed to a fixed reference pressure p_r in an isentropic and isohaline manner. Because heat fluxes into and out of the ocean occur mostly near the sea surface, the reference pressure for potential enthalpy is always taken to be $p_r = 0$ dbar (that is, at zero sea pressure). Potential enthalpy can be expressed as the pressure integral of specific volume as (from McDougall (2003) and see the discussion below Eqn. (2.8.2))

$$\begin{aligned}
 h^0(S_A, t, p) &= h(S_A, \theta, 0) = \tilde{h}^0(S_A, \theta) = h(S_A, t, p) - \int_{p_0}^p v(S_A, \theta[S_A, t, p, p'], p') dP' \\
 &= h(S_A, t, p) - \int_{p_0}^p \hat{v}(S_A, \eta, p') dP' \\
 &= h(S_A, t, p) - \int_{p_0}^p \tilde{v}(S_A, \theta, p') dP' \\
 &= h(S_A, t, p) - \int_{p_0}^p \hat{v}(S_A, \Theta, p') dP',
 \end{aligned} \tag{3.2.1}$$

and we emphasize that the pressure integrals here must be done with respect to pressure expressed in Pa rather than dbar. In terms of the Gibbs function, potential enthalpy h^0 is evaluated as

$$h^0(S_A, t, p) = h(S_A, \theta, 0) = g(S_A, \theta, 0) - (T_0 + \theta)g_T(S_A, \theta, 0). \tag{3.2.2}$$

Also, note that dynamic enthalpy is defined as enthalpy minus potential enthalpy (Young, 2010) and is available as the function **gsw_dynamic_enthalpy** in the GSW Toolbox.

3.3 Conservative Temperature

Conservative Temperature Θ is defined to be proportional to potential enthalpy,

$$\Theta(S_A, t, p) = \tilde{\Theta}(S_A, \theta) = h^0(S_A, t, p)/c_p^0 = \tilde{h}^0(S_A, \theta)/c_p^0 \quad (3.3.1)$$

where the value that is chosen for c_p^0 is motivated in terms of potential enthalpy evaluated at an Absolute Salinity of $S_{SO} = 35u_{PS} = 35.165\ 04\ \text{g kg}^{-1}$ and at $\theta = 25\ ^\circ\text{C}$ by

$$\frac{[h(S_{SO}, 25^\circ\text{C}, 0) - h(S_{SO}, 0^\circ\text{C}, 0)]}{(25\ \text{K})} \approx 3991.867\ 957\ 119\ 63\ \text{J kg}^{-1}\ \text{K}^{-1}, \quad (3.3.2)$$

noting that $h(S_{SO}, 0\ ^\circ\text{C}, 0\ \text{dbar})$ is zero according to the way the Gibbs function is defined in (2.6.5). We adopt the exact definition for c_p^0 to be the 15-digit value in (3.3.2), so that

$$c_p^0 \equiv 3991.867\ 957\ 119\ 63\ \text{J kg}^{-1}\ \text{K}^{-1}. \quad (3.3.3)$$

When IAPWS-95 is used for the pure water part of the Gibbs function, $\Theta(S_{SO}, 0\ ^\circ\text{C}, 0)$ and $\Theta(S_{SO}, 25\ ^\circ\text{C}, 0)$ differ from $0\ ^\circ\text{C}$ and $25\ ^\circ\text{C}$ respectively by the round-off amount of $5 \times 10^{-12}\ ^\circ\text{C}$. When IAPWS-09 (which is based on the paper of Feistel (2003), see appendix G) is used for the pure water part of the Gibbs function, $\Theta(S_{SO}, 0\ ^\circ\text{C}, 0)$ differs from $0\ ^\circ\text{C}$ by $-8.25 \times 10^{-8}\ ^\circ\text{C}$ and $\Theta(S_{SO}, 25\ ^\circ\text{C}, 0)$ differs from $25\ ^\circ\text{C}$ by $9.3 \times 10^{-6}\ ^\circ\text{C}$. Over the temperature range from $0\ ^\circ\text{C}$ to $40\ ^\circ\text{C}$ the difference between Conservative Temperature using IAPWS-95 and IAPWS-09 as the pure water part is no more than $\pm 1.5 \times 10^{-5}\ ^\circ\text{C}$, a temperature difference that will be ignored.

The value of c_p^0 in (3.3.3) is very close to the average value of the specific heat capacity c_p at the sea surface of today's global ocean. This value of c_p^0 also causes the average value of $\theta - \Theta$ at the sea surface to be very close to zero. Since c_p^0 is simply a constant of proportionality between potential enthalpy and Conservative Temperature, it is totally arbitrary, and we see no reason why its value would need to change from (3.3.3) even when in future decades an improved Gibbs function of seawater is agreed upon.

Appendix A.18 outlines why Conservative Temperature gets its name; it is approximately two orders of magnitude more conservative compared with either potential temperature or entropy.

The SIA and GSW software libraries both include an algorithm for determining Conservative Temperature Θ from values of Absolute Salinity S_A and potential temperature θ referenced to $p = 0\ \text{dbar}$. These libraries also have an algorithm for evaluating potential temperature (referenced to $0\ \text{dbar}$) from S_A and Θ . This inverse algorithm, $\hat{\theta}(S_A, \Theta)$, has an initial seed based on a rational function approximation and finds potential temperature to machine precision ($\sim 10^{-14}\ ^\circ\text{C}$) in one and a half iterations of a modified Newton-Raphson technique (McDougall *et al.* (2011b)).

A.30 Computationally efficient 75-term expression for the specific volume of seawater in terms of Θ

Ocean models to date have treated their salinity and temperature variables as being Practical Salinity S_p and potential temperature θ . Ocean models that are TEOS-10 compatible need to calculate Absolute Salinity S_A and Conservative Temperature Θ (as discussed in appendices A.20 and A.21), and they need a computationally efficient expression for calculating specific volume (or density) in terms of Absolute Salinity S_A , Conservative Temperature Θ and pressure p .

Following the work of McDougall *et al.* (2003) and Jackett *et al.* (2006), the TEOS-10 specific volume \hat{v} has been approximated by a 75-term polynomial by Roquet *et al.* (2015). This polynomial is expressed in terms of the following three dimensionless salinity, temperature and pressure variables,

$$s \equiv \sqrt{\frac{S_A + 24 \text{ g kg}^{-1}}{S_{Au}}}, \quad \tau \equiv \frac{\Theta}{\Theta_u} \quad \text{and} \quad \pi \equiv \frac{p}{p_u}, \quad (\text{A.30.1})$$

in terms of the unit-related scaling constants

$$S_{Au} \equiv 40 \times 35.16504 \text{ g kg}^{-1} / 35, \quad \Theta_u \equiv 40^\circ\text{C} \quad \text{and} \quad p_u \equiv 10^4 \text{ dbar}. \quad (\text{A.30.2})$$

Their polynomial expression for the specific volume of seawater is

$$\hat{v}(S_A, \Theta, p) = v_u \sum_{i,j,k} v_{ijk} s^i \tau^j \pi^k, \quad (\text{A.30.3})$$

where $v_u \equiv 1 \text{ m}^3 \text{ kg}^{-1}$ and the non-zero dimensionless constants v_{ijk} are given in Table K.1 of appendix K. The specific volume data was fitted in a “funnel” of data points in (S_A, Θ, p) space (McDougall *et al.* (2003)) which extends to a pressure of 8000 dbar. At the sea surface the “funnel” covers the full range of temperature and salinity while for pressures greater than 6500 dbar the maximum temperature of the fitted data is 10°C and the minimum Absolute Salinity is 30 g kg^{-1} . That is, the fit has been performed over a region of parameter space which includes water that is approximately 8°C warmer and 5 g kg^{-1} fresher in the deep ocean than the seawater which exists in the present ocean.

As outlined in appendix K, this 75-term polynomial expression for v yields the thermal expansion and saline contraction coefficients, α^Θ and β^Θ , that are essentially as accurate as those derived from the full TEOS-10 Gibbs function for data in the “oceanographic funnel”. In dynamical oceanography it is these thermal expansion and haline contraction coefficients which are the most important aspects of the equation of state since the “thermal wind” is proportional to $\alpha^\Theta \nabla_p \Theta - \beta^\Theta \nabla_p S_A$ and the vertical static stability is given in terms of the buoyancy frequency N by $g^{-1} N^2 = \alpha^\Theta \Theta_z - \beta^\Theta (S_A)_z$. Hence for dynamical oceanography we may take Roquet *et al.*'s (2015) 75-term polynomial expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

Appendix P describes how an expression for the enthalpy of seawater in terms of Conservative Temperature, specifically the functional form $\hat{h}(S_A, \Theta, p)$, together with an expression for entropy in the form $\hat{\eta}(S_A, \Theta)$, can be used as an alternative thermodynamic potential to the Gibbs function $g(S_A, t, p)$. The need for the functional form $\hat{h}(S_A, \Theta, p)$ also arises in section 3.32 and in Eqns. (3.26.3) and (3.29.1). The 75-term expression, Eqn. (A.30.3) for $v^{75} = \hat{v}^{75}(S_A, \Theta, p)$ can be used to find a closed expression for $\hat{h}(S_A, \Theta, p)$ by integrating $\hat{v}^{75}(S_A, \Theta, p)$ with respect to pressure (in Pa), since $\hat{h}_p = v = \rho^{-1}$ (see Eqn. (2.8.3)). Specific enthalpy calculated from $\hat{v}^{75}(S_A, \Theta, p)$ is available in the GSW Oceanographic Toolbox as the function `gsw_enthalpy(SA,CT,p)`. Using `gsw_enthalpy` to evaluate $\hat{h}(S_A, \Theta, p)$ is 7 times faster than first evaluating the in situ temperature t (from `gsw_t_from_CT(SA,CT,p)`) and then calculating enthalpy from the full Gibbs function

expression $h(S_A, t, p)$ using **gsw_enthalpy_t_exact**(SA,t,p). (These last two function calls have also been combined into the one function, **gsw_enthalpy_CT_exact**(SA,CT,p).)

Also, the enthalpy difference at the same values of S_A and Θ but at different pressures (see Eqn. (3.32.5)) is available as the function **gsw_enthalpy_diff**(SA,CT,p_shallow,p_deep).

Following Young (2010), the difference between h and $c_p^0 \Theta$ is called “dynamic enthalpy” and can be found using the function **gsw_dynamic_enthalpy**(SA,CT,p) in the GSW Oceanographic Toolbox.

Appendix K: Coefficients of the 75-term expression for the specific volume of seawater in terms of Θ

The TEOS-10 Gibbs function of seawater $g(S_A, t, p)$ is written as a polynomial in terms of in situ temperature t , while for ocean models, specific volume (or density) needs to be expressed as a computationally efficient expression in terms of Conservative Temperature Θ . Roquet *et al.* (2015) have published such a computationally efficient polynomial for specific volume. Their non-dimensional (root) salinity s , temperature τ , and pressure π , variables are

$$s \equiv \sqrt{\frac{S_A + 24 \text{ g kg}^{-1}}{S_{A_u}}}, \quad \tau \equiv \frac{\Theta}{\Theta_u} \quad \text{and} \quad \pi \equiv \frac{p}{p_u}, \quad (\text{K.1})$$

in terms of the unit-related scaling constants

$$S_{A_u} \equiv 40 \times 35.16504 \text{ g kg}^{-1} / 35, \quad \Theta_u \equiv 40^\circ\text{C} \quad \text{and} \quad p_u \equiv 10^4 \text{ dbar}. \quad (\text{K.2})$$

Their polynomial expression for the specific volume of seawater is

$$\hat{v}(S_A, \Theta, p) = v_u \sum_{i,j,k} v_{ijk} s^i \tau^j \pi^k, \quad (\text{K.3})$$

where $v_u \equiv 1 \text{ m}^3 \text{ kg}^{-1}$ and the non-zero dimensionless constants v_{ijk} are given in Table K.1.

Roquet *et al.* (2015) fitted the TEOS-10 values of specific volume v to S_A , Θ and p in a “funnel” of data points in (S_A, Θ, p) space. This is the same “funnel” of data points as used in McDougall *et al.* (2003); at the sea surface it covers the full range of temperature and salinity while for pressure greater than 6500 dbar, the maximum temperature of the fitted data is 10°C and the minimum Absolute Salinity is 30 g kg^{-1} . The maximum pressure of the “funnel” is 8000 dbar. Table K.1 contains the 75 coefficients of the expression (K.3) for specific volume in terms of (S_A, Θ, p) .

The rms error of this 75-term approximation to the full Gibbs function-derived TEOS-10 specific volume over the “funnel” is $0.2 \times 10^{-9} \text{ m}^3 \text{ kg}^{-1}$; this can be compared with the rms uncertainty of $4 \times 10^{-9} \text{ m}^3 \text{ kg}^{-1}$ of the underlying laboratory density data to which the TEOS-10 Gibbs function was fitted (see the first two rows of Table O.1 of appendix O). Similarly, the appropriate thermal expansion coefficient,

$$\alpha^\Theta = \frac{1}{v} \left. \frac{\partial v}{\partial \Theta} \right|_{S_A, p} = - \frac{1}{\rho} \left. \frac{\partial \rho}{\partial \Theta} \right|_{S_A, p}, \quad (\text{K.4})$$

of the 75-term equation of state is different from the same thermal expansion coefficient evaluated from the full Gibbs function-derived TEOS-10 with an rms error in the “funnel” of $0.03 \times 10^{-6} \text{ K}^{-1}$; this can be compared with the rms error of the thermal expansion coefficient of the laboratory data to which the Feistel (2008) Gibbs function was fitted of $0.73 \times 10^{-6} \text{ K}^{-1}$ (see row six of Table O.1 of appendix O). In terms of the evaluation of density gradients, the haline contraction coefficient evaluated from Eqn. (K.3) is many times more accurate than the thermal expansion coefficient. Hence we may consider the 75-term polynomial expression for specific volume, Eqn. (K.3), to be equally as accurate as the full TEOS-10 expressions for specific volume, for the thermal expansion coefficient and for the haline contraction coefficient for data that reside inside the “oceanographic funnel”.

The sound speed evaluated from the 75-term polynomial of Eqn. (K.3) has an rms error over the “funnel” of 0.025 m s^{-1} which is a little less than the rms error of the underlying sound speed data that was incorporated into the Feistel (2008) Gibbs function, being 0.035 m s^{-1} (see rows 7 to 9 of Table O.1 of appendix O). Hence, especially for the purposes of dynamical oceanography where α^Θ and β^Θ are the aspects of the equation of

state that, together with spatial gradients of S_A and Θ , drive ocean currents and affect the calculation of the buoyancy frequency, we may take the 75-term expression for specific volume, Eqn. (K.3), as essentially reflecting the full accuracy of TEOS-10.

The use of Eqn. (K.3) to evaluate $\hat{v}(S_A, \Theta, p)$ or $\hat{\rho}(S_A, \Theta, p)$ from `gsw_specvol(SA,CT,p)` or `gsw_rho(SA,CT,p)` is approximately five times faster than first evaluating the *in situ* temperature t (from `gsw_t_from_CT(SA,CT,p)`) and then calculating *in situ* specific volume or density from the full Gibbs function expression $v(S_A, t, p)$ or $\rho(S_A, t, p)$ via `gsw_specvol_t_exact(SA,t,p)` or `gsw_rho_t_exact(SA,t,p)`. (These two function calls have been combined into `gsw_specvol_CT_exact(SA,CT,P)` and `gsw_rho_CT_exact(SA,CT,P)`.)

Table K.1. Coefficients of the 75-term polynomial of Roquet *et al.* (2015).

i	j	k	v_{ijk}	i	j	k	v_{ijk}	i	j	k	v_{ijk}
0	0	0	1.0769995862e-3	0	5	0	-8.0539615540e-7	1	0	2	-5.8484432984e-7
1	0	0	-3.1038981976e-4	1	5	0	-3.3052758900e-7	2	0	2	-4.8122251597e-6
2	0	0	6.6928067038e-4	0	6	0	2.0543094268e-7	3	0	2	4.9263106998e-6
3	0	0	-8.5047933937e-4	0	0	1	-6.0799143809e-5	4	0	2	-1.7811974727e-6
4	0	0	5.8086069943e-4	1	0	1	2.4262468747e-5	0	1	2	-1.1736386731e-6
5	0	0	-2.1092370507e-4	2	0	1	-3.4792460974e-5	1	1	2	-5.5699154557e-6
6	0	0	3.1932457305e-5	3	0	1	3.7470777305e-5	2	1	2	5.4620748834e-6
0	1	0	-1.5649734675e-5	4	0	1	-1.7322218612e-5	3	1	2	-1.3544185627e-6
1	1	0	3.5009599764e-5	5	0	1	3.0927427253e-6	0	2	2	2.1305028740e-6
2	1	0	-4.3592678561e-5	0	1	1	1.8505765429e-5	1	2	2	3.9137387080e-7
3	1	0	3.4532461828e-5	1	1	1	-9.5677088156e-6	2	2	2	-6.5731104067e-7
4	1	0	-1.1959409788e-5	2	1	1	1.1100834765e-5	0	3	2	-4.6132540037e-7
5	1	0	1.3864594581e-6	3	1	1	-9.8447117844e-6	1	3	2	7.7618888092e-9
0	2	0	2.7762106484e-5	4	1	1	2.5909225260e-6	0	4	2	-6.3352916514e-8
1	2	0	-3.7435842344e-5	0	2	1	-1.1716606853e-5	0	0	3	-1.1309361437e-6
2	2	0	3.5907822760e-5	1	2	1	-2.3678308361e-7	1	0	3	3.6310188515e-7
3	2	0	-1.8698584187e-5	2	2	1	2.9283346295e-6	2	0	3	1.6746303780e-8
4	2	0	3.8595339244e-6	3	2	1	-4.8826139200e-7	0	1	3	-3.6527006553e-7
0	3	0	-1.6521159259e-5	0	3	1	7.9279656173e-6	1	1	3	-2.7295696237e-7
1	3	0	2.4141479483e-5	1	3	1	-3.4558773655e-6	0	2	3	2.8695905159e-7
2	3	0	-1.4353633048e-5	2	3	1	3.1655306078e-7	0	0	4	1.0531153080e-7
3	3	0	2.2863324556e-6	0	4	1	-3.4102187482e-6	1	0	4	-1.1147125423e-7
0	4	0	6.9111322702e-6	1	4	1	1.2956717783e-6	0	1	4	3.1454099902e-7
1	4	0	-8.7595873154e-6	0	5	1	5.0736766814e-7	0	0	5	-1.2647261286e-8
2	4	0	4.3703680598e-6	0	0	2	9.9856169219e-6	0	0	6	1.9613503930e-9