3.31 Pressure-integrated steric height

The depth-integrated mass flux of the geostrophic Eulerian flow between two fixed pressure levels can also be represented by a streamfunction. Using the hydrostatic relation $P_z = -g\rho$, and assuming the gravitational acceleration to be independent of height, the depth-integrated mass flux $\int \rho \mathbf{v}\, dz$ is given by $-g^{-1}\int \mathbf{v}\, dP$ and this motivates taking the pressure integral of the Dynamic Height Anomaly Ψ (from Eqn. (3.27.1)) to form the Pressure-Integrated-Steric-Height $PISH$ (also called Depth-Integrated Steric Height $DISH$ by Godfrey (1989)),

$$\begin{aligned} PISH \;=\; \Psi' \;=\; -g^{-1}\int_{P_0}^{P} \Psi\left(p''\right) dP'' \;&=\; g^{-1}\int_{P_0}^{P}\int_{P_0}^{P''} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right], \Theta\left[p'\right], p'\right) dP'\, dP'' \\ &=\; g^{-1}\int_{P_0}^{P} \left(P - P'\right) \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right], \Theta\left[p'\right], p'\right) dP'. \end{aligned} \qquad (3.31.1)$$

The two-dimensional gradient of Ψ' is related to the depth-integrated mass flux of the velocity difference with respect to the velocity at zero sea pressure, $\mathbf{v}_0$, according to

$$\mathbf{k}\times\nabla_p\Psi' = f\int_{z(P)}^{z(P_0)}\rho\left[\mathbf{v}(z') - \mathbf{v}_0\right]dz' = g^{-1}f\int_{P_0}^{P}\left[\mathbf{v}(p') - \mathbf{v}_0\right]dP'. \tag{3.31.2}$$

The definition, Eqn. (3.31.1), of $PISH$ applies to all choices of the reference values $\hat{\hat{S}}_\mathrm{A}$, $\tilde{\tilde{S}}_\mathrm{A}$ and $\hat{\hat{t}}$, $\tilde{\tilde{\theta}}$ or $\tilde{\tilde{\Theta}}$ in the definitions, Eqns. (3.7.2 – 3.7.4), of the specific volume anomaly.

Since the velocity at depth in the ocean is generally much smaller than at the sea surface, it is customary to take the reference pressure to be some constant (deep) pressure P_1 so that Eqn. (3.27.1) becomes

$$\Psi = \int_{P}^{P_1}\hat{\delta}\left(S_\mathrm{A}\left[p'\right],\Theta\left[p'\right],p'\right)dP', \tag{3.31.3}$$

and $PISH$, reflecting the depth-integrated horizontal mass transport from the sea surface to pressure P_1, relative to the flow at P_1, is

$$\begin{aligned} PISH = \Psi' &= g^{-1}\int_{P_0}^{P_1}\Psi\left(p''\right)dP'' = g^{-1}\int_{P_0}^{P_1}\int_{P''}^{P_1}\hat{\delta}\left(S_\mathrm{A}\left[p'\right],\Theta\left[p'\right],p'\right)dP'\,dP'' \\ &= g^{-1}\int_{P_0}^{P_1}\left(P'-P_0\right)\hat{\delta}\left(S_\mathrm{A}\left[p'\right],\Theta\left[p'\right],p'\right)dP' \\ &= \tfrac{1}{2}g^{-1}\int_{0}^{\left(P_1-P_0\right)^2}\hat{\delta}\left(S_\mathrm{A}\left[p'\right],\Theta\left[p'\right],p'\right)\,d\left(\left(P'-P_0\right)^2\right). \end{aligned} \tag{3.31.4}$$

The two-dimensional gradient of Ψ' is now related to the depth-integrated mass flux of the velocity difference with respect to the velocity at P_1, $\mathbf{v}_1$, according to

$$\mathbf{k}\times\nabla_p\Psi' = f\int_{z(P_1)}^{z(P_0)}\rho\left[\mathbf{v}(z') - \mathbf{v}_1\right]dz' = g^{-1}f\int_{P_0}^{P_1}\left[\mathbf{v}(p') - \mathbf{v}_1\right]dP'. \tag{3.31.5}$$

The specific volume anomaly $\hat{\delta}$ in Eqns. (3.31.1), (3.31.3) and (3.31.4) can be replaced with specific volume v without affecting the isobaric gradient of the resulting streamfunction. That is, this substitution in Ψ' does not affect Eqn. (3.31.2) or Eqn. (3.31.5), as the additional term is a function only of pressure. With specific volume in place of specific volume anomaly, Eqn. (3.31.4) becomes the depth-integrated gravitational potential energy of the water column (plus a very small term that is present because the atmospheric pressure is not zero, McDougall *et al.* (2003)).

$PISH$ should be quoted in units of $\mathrm{kg\ s^{-2}}$ so that its two-dimensional gradient has the same units as the depth-integrated flux of $\rho\left[\mathbf{v}\left(z'\right) - \mathbf{v}_1\right]$ times the Coriolis frequency.