Notes on the function gsw_geo_stf_Montgomery(SA,CT,p,p_ref)

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This function, **gsw_geo_strf_Montgomery**(SA,CT,p,p_ref) evaluates the Montgomery geostrophic streamfunction (Montgomery, 1937) $\Psi^{\mathrm{M}}\left(S_{\mathrm{A}},\, \Theta,\, p,\, p_{\mathrm{ref}}\right)$ using the the computationally-efficient 75-term polynomial expression for specific volume $\hat{v}\left(S_{\mathrm{A}},\Theta, p\right)$ of Roquert et al. (2015) in terms of Absolute Salinity S_{A}, Conservative Temperature Θ and pressure p. This 75-term polynomial expression for specific volume is discussed in Roquert et al. (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al*. (2010)). For dynamical oceanography we may take the 75-term polynomial for specific volume as essentially reflecting the full accuracy of TEOS-10.

The input variables SA,CT,p are either a single vertical cast or a series of such vertical casts. The input variable p_{ref} is a single positive scalar reference pressure (in dbar). When p_ref is zero, **gsw_geo_strf_Montgomery**(SA,CT,p,p_ref) returns the Montgomery geostrophic streamfunction with respect to the sea surface, otherwise, the function returns the Montgomery geostrophic streamfunction with respect to the (deep) reference pressure p_ref. The Montgomery geostrophic streamfunction $\Psi^{\mathrm{M}} = \Psi^{\mathrm{M}}\left(S_{\mathrm{A}},\, \Theta,\, p,\, p_{\mathrm{ref}}\right)$ is designed to be used in a surface of constant specific volume anomaly, defined in Eqn. (2) below, that is, in a surface in which $\hat{\delta}\left(S_{\mathrm{A}},\Theta, p\right)$ is a constant, $\hat{\delta}\left(S_{\mathrm{A}},\Theta, p\right) = \hat{\delta}_1$. The geostrophic streamfunction Ψ^{M} is the geostrophic streamfunction for the flow in the specific volume anomaly surface $\hat{\delta}\left(S_{\mathrm{A}},\Theta, p\right) = \hat{\delta}_1$ relative to the flow at $P = P_{\mathrm{ref}}$ (that is, at p_{ref} dbar). Thus the two-dimensional gradient of Ψ^{M} in the $\hat{\delta}_1$ specific volume anomaly surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ in the $\hat{\delta} = \hat{\delta}_1$ surface and that at the reference pressure $\mathbf{v}_{\mathrm{ref}}$ according to

$$\mathbf{k}\times\nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} \;=\; f\mathbf{v} - f\mathbf{v}_{\mathrm{ref}} \qquad \text{or} \qquad \nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} \;=\; -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_{\mathrm{ref}}\right). \tag{1}$$

The specific volume anomaly $\hat{\delta}\left(S_{\mathrm{A}},\Theta, p\right)$ is defined with respect to the constant reference values $S_{\mathrm{SO}} \equiv 35.165\,04\ \mathrm{g\ kg^{-1}}$ and $\Theta = 0°\mathrm{C}$ as

$$\hat{\delta}\left(S_{\mathrm{A}},\Theta, p\right) \;=\; \hat{v}\left(S_{\mathrm{A}},\Theta, p\right) - \hat{v}\left(S_{\mathrm{SO}},0°\mathrm{C}, p\right), \tag{2}$$

and the Montgomery geostrophic streamfunction Ψ^{M} is defined in terms of the dynamic height anomaly Ψ by

$$\begin{aligned} \Psi^{\mathrm{M}} \;&=\; \left(P - P_0\right)\hat{\delta} \;-\; \int_{P_{\mathrm{ref}}}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right], p'\right) dP' \\ &=\; \left(P - P_0\right)\hat{\delta} \;+\; \Psi \end{aligned} \tag{3}$$

Note also that the pressure integral in Eqn. (3) is done with pressure increments measured in Pa, not dbar. This ensures that specific volume and the Montgomery geostrophic streamfunction retain their usual units of $\mathrm{m^3\ kg^{-1}}$ and $\mathrm{m^2\ s^{-2}}$ ($=\mathrm{J\ kg^{-1}}$) respectively. The code **gsw_geo_strf_Montgomery**(SA,CT,p,p_ref) operates by evaluating the last line of Eqn. (3) and the dynamic height anomaly Ψ is found from **gsw_geo_strf_dyn_height**(SA,CT,p,p_ref).

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org See section 3.27 and appendix A.30.

Montgomery, R. B., 1937. A suggested method for representing gradient flow in isentropic surfaces. *Bull. Amer. Meteor. Soc.* **18**, 210–212.

Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. Ocean Modelling, 90, pp. 29-43. http://dx.doi.org/10.1016/j.ocemod.2015.04.002

Here follows section 3.28 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.28 Montgomery geostrophic streamfunction

The Montgomery "acceleration potential" (Montgomery, 1937) Ψ^{M} defined by

$$\Psi^{\mathrm{M}} = \left(P - P_0\right)\hat{\delta} - \int_{P_0}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right],p'\right) dP' = \left(P - P_0\right)\hat{\delta} + \Psi \tag{3.28.1}$$

is the geostrophic streamfunction for the flow in the specific volume anomaly surface $\hat{\delta}\left(S_{\mathrm{A}},\Theta,p\right) = \hat{\delta}_1$ relative to the flow at $P = P_0$ (that is, at $p = 0\,\mathrm{dbar}$). Thus the two-dimensional gradient of Ψ^{M} in the $\hat{\delta}_1$ specific volume anomaly surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ in the $\hat{\delta} = \hat{\delta}_1$ surface and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k}\times\nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} = f\mathbf{v} - f\mathbf{v}_0 \qquad \text{or} \qquad \nabla_{\hat{\delta}_1}\Psi^{\mathrm{M}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right). \tag{3.28.2}$$

The definition, Eqn. (3.28.1), of the Montgomery geostrophic streamfunction applies to all choices of the reference values $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ in the definition, Eqn. (3.7.3), of the specific volume anomaly. By carefully choosing these reference values the specific volume anomaly surface can be made to closely approximate the neutral tangent plane (McDougall and Jackett (2007)).

It is not uncommon to read of authors using the Montgomery geostrophic streamfunction, Eqn. (3.28.1), as a geostrophic streamfunction in surfaces other than specific volume anomaly surfaces. This incurs errors that should be recognized. For example, the gradient of the Montgomery geostrophic streamfunction, Eqn. (3.28.1), in a neutral tangent plane becomes (instead of Eqn. (3.28.2) in the $\hat{\delta} = \hat{\delta}_1$ surface)

$$\nabla_n\Psi^{\mathrm{M}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right) + \left(P - P_0\right)\nabla_n\hat{\delta}\,, \tag{3.28.3}$$

where the last term represents an error arising from using the Montgomery streamfunction in a surface other than the surface for which it was derived.

Zhang and Hogg (1992) subtracted an arbitrary pressure offset, $\left(\bar{P} - P_0\right)$, from $\left(P - P_0\right)$ in the first term in Eqn. (3.28.1), so defining the modified Montgomery streamfunction

$$\Psi^{\text{Z-H}} = \left(P - \bar{P}\right)\hat{\delta} - \int_{P_0}^{P} \hat{\delta}\left(S_{\mathrm{A}}\left[p'\right],\Theta\left[p'\right],p'\right) dP'. \tag{3.28.4}$$

The gradient of $\Psi^{\text{Z-H}}$ in a neutral tangent plane becomes

$$\nabla_n\Psi^{\text{Z-H}} = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right) + \left(P - \bar{P}\right)\nabla_n\hat{\delta}\,, \tag{3.28.5}$$

where the last term can be made significantly smaller than the corresponding term in Eqn. (3.28.3) by choosing the constant pressure $\bar{P}$ to be close to the average pressure on the surface. This term can be further minimized by suitably choosing the constant reference values $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ in the definition, Eqn. (3.7.3), of specific volume anomaly $\tilde{\tilde{\delta}}$ so that this surface more closely approximates the neutral tangent plane (McDougall (1989)). This improvement is available because it can be shown that

$$\rho \nabla_n \tilde{\tilde{\delta}} \;=\; -\left[\hat{\kappa}\left(S_{\mathrm{A}}, \Theta, p\right) - \hat{\kappa}\left(\tilde{\tilde{S}}_{\mathrm{A}}, \tilde{\tilde{\Theta}}, p\right)\right] \nabla_n P \;\approx\; T_b^{\Theta}\left(\Theta - \tilde{\tilde{\Theta}}\right) \nabla_n P. \qquad (3.28.6)$$

The last term in Eqn. (3.28.5) is then approximately

$$\left(P - \bar{P}\right) \nabla_n \tilde{\tilde{\delta}} \;\approx\; \tfrac{1}{2} \rho^{-1} T_b^{\Theta}\left(\Theta - \tilde{\tilde{\Theta}}\right) \nabla_n \left(P - \bar{P}\right)^2 \qquad (3.28.7)$$

and hence suitable choices of $\bar{P}$, $\tilde{\tilde{S}}_{\mathrm{A}}$ and $\tilde{\tilde{\Theta}}$ can reduce the last term in Eqn. (3.28.5) that represents the error in interpreting the Montgomery geostrophic streamfunction, Eqn. (3.28.4), as the geostrophic streamfunction in a surface that is more neutral than a specific volume anomaly surface.

The Montgomery geostrophic streamfunction should be quoted in units of $\mathrm{m^2\ s^{-2}}$. These are the units in which the GSW Toolbox outputs the Montgomery geostrophic streamfunction in the function **gsw_geo_strf_Montgomery**(SA,CT,p,p_ref). When the last argument of this function, p_ref, is other than zero, the function returns the Montgomery geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to $p = 0$ dbar (i.e. $P = P_0$) as in Eqn. (3.28.1).