Notes on the function
gsw_geo_strf_isopycnal_pc(SA,CT,delta_p,Neutral_Density,layer_index,A)

Notes made 23th September 2010, updated 19th April 2011

In density-coordinate forward ocean models, the variables are usually interpreted as being piecewise constant in the vertical, and this function, **gsw_geo_strf_isopycnal_pc**, calculates the approximate "isopycnal" geostrophic streamfunction of McDougall and Klocker (2010) taking this piecewise-constant nature of Absolute Salinity and Conservative Temperature into account. The code uses the 75-term polynomial function expression for specific volume **gsw_specvol**(SA,CT,p). This 75-term polynomial expression for specific volume is discussed in Roquert *et al*. (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al*. (2010)).For dynamical oceanography we may take the 75-term polynomial expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

In density-coordinate models (often called "isopycnal models") the "thickness" of the *i*th layer, $p^{i+1} - p^{i}$, varies with latitude, longitude and time. In these layered models the "thicknesses" of the least dense layers are often zero at many locations in the ocean, because these density surfaces have outcropped.

The McDougall-Klocker geostrophic streamfunction is defined by

$$\varphi^{\mathrm{n}} = \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\tilde{\tilde{\delta}}\left(S_{\mathrm{A}},\Theta,p\right) - \tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\left(P-\tilde{\tilde{P}}\right)^{2} - \int_{P_0}^{P}\tilde{\tilde{\delta}}\,dP'\,, \tag{1}$$

where $\rho^{-1}T_{\mathrm{b}}^{\Theta}$ is taken to be the constant value $2.7x10^{-15}\,\mathrm{K}^{-1}(\mathrm{Pa})^{-2}\mathrm{m}^{2}\mathrm{s}^{-2}$. This geostrophic streamfunction is for use in approximately neutral surfaces, such as an ω-surface of Klocker *et al.* (2009a,b) or a Neutral Density (γ^{n}) surface of Jackett and McDougall (1997)) or a suitably referenced potential density surface such as a σ_2 surface. A suitable reference seawater parcel $\left(\tilde{\tilde{S}}_{\mathrm{A}},\tilde{\tilde{\Theta}},\tilde{\tilde{p}}\right)$ is selected for each approximately neutral surface that one is considering, and the specific volume anomaly $\tilde{\tilde{\delta}}$ is defined as

$$\tilde{\tilde{\delta}}\left(S_{\mathrm{A}},\Theta,p\right) = \hat{v}\left(S_{\mathrm{A}},\Theta,p\right) - \hat{v}\left(\tilde{\tilde{S}}_{\mathrm{A}},\tilde{\tilde{\Theta}},p\right). \tag{2}$$

In this function, **gsw_geo_strf_isopycnal_pc**, the reference values $\left(\tilde{\tilde{S}}_{\mathrm{A}},\tilde{\tilde{\Theta}},\tilde{\tilde{p}}\right)$ are found by interpolation down a single reference cast which includes these three variables as well as values of Neutral Density γ^{n} and also of σ_2. This reference cast is interpolated with respect to γ^{n}, to the value of γ^{n} which is supplied by the user of this function, with one value of γ^{n} for each "density" surface that is selected. If this function is being used with a series of approximately neutral surfaces which are neither Neutral Density surfaces nor ω-surfaces, then a value of γ^{n} needs to be assigned to each of the user's surfaces. This could be achieved, for example, by the user labeling a vertical profile of her/his data from the mid equatorial Pacific with Neutral Density and associating these values of Neutral Density to the whole of the user's layers; with one value of γ^{n} for each selected approximately neutral surface. As an alternative to having to label each layer with Neutral Density, we have also allowed for the option of having the reference data set to be interpolated with respect to σ_2, the potential density with respect to 2000 dbar. In this case, the user needs to supply the σ_2 value for each layer on which the McDougall-Klocker streamfunction is required.

The last term in Eqn. (1) is obtained with the aid of a call to **gsw_geo_strf_dyn_height_pc** which returns the dynamic height anomaly Ψ for the piecewise constant vertical profile, with respect to $S_{\mathrm{SO}} \equiv 35.165\,04\ \mathrm{g\ kg}^{-1}$ and $\Theta = 0°\mathrm{C}$. The last term in Eqn. (1) becomes

$$-\int_{P_0}^{P} \tilde{\tilde{\delta}}\, dP' = \Psi + \hat{h}\Big(\tilde{\tilde{S}}_{\mathrm{A}}, \tilde{\tilde{\Theta}}, \tfrac{1}{2}\big[p^{n+1} + p^{n}\big]\Big) - c_p^0 \tilde{\tilde{\Theta}} - \hat{h}\Big(S_{\mathrm{SO}}, \Theta = 0°\mathrm{C}, \tfrac{1}{2}\big[p^{n+1} + p^{n}\big]\Big) \tag{3}$$

where $\tfrac{1}{2}\big[p^{n+1} + p^{n}\big]$ is the pressure at the mid-pressure of the nth layer. The second term in Eqn. (3) is obtained from a call to **gsw_enthalpy** and the last term via a call to **gsw_enthalpy_SSO_0**(p) which is simply a streamlined version of **gsw_enthalpy**. Combining Eqns. (1) and (3), the expression that is used to evaluate **gsw_geo_strf_isopycnal_pc** is

$$\begin{aligned} \varphi^{\mathrm{n}} = {} & \tfrac{1}{2}\big(P - \tilde{\tilde{P}}\big)\tilde{\tilde{\delta}}\big(S_{\mathrm{A}}, \Theta, p\big) - \tfrac{1}{12}\rho^{-1} T_{\mathrm{b}}^{\Theta}\big(\Theta - \tilde{\tilde{\Theta}}\big)\big(P - \tilde{\tilde{P}}\big)^{2} + \Psi \\ & + \hat{h}\Big(\tilde{\tilde{S}}_{\mathrm{A}}, \tilde{\tilde{\Theta}}, \tfrac{1}{2}\big[p^{n+1} + p^{n}\big]\Big) - c_p^0 \tilde{\tilde{\Theta}} - \hat{h}\Big(S_{\mathrm{SO}}, \Theta = 0°\mathrm{C}, \tfrac{1}{2}\big[p^{n+1} + p^{n}\big]\Big), \end{aligned} \tag{4}$$

where Ψ is the output of **gsw_geo_strf_dyn_height_pc**.

The pressure input to the function **gsw_geo_strf_isopycnal_pc** is the matrix delta_p = $p^{i+1} - p^{i}$ and the code can handle zero values of this layer "thickness" in several layers (which occurs when density surface outcrop). The function calculates the pressures p_mid at the mid-pressure of each layer by summing the delta_p of the shallower layers and adding half of delta_p = $p^{i+1} - p^{i}$ at the ith layer (this summation is performed by **gsw_geo_strf_dyn_height_pc** and returned to the present function).

In **gsw_geo_strf_isopycnal_pc**(SA,CT,delta_p,Neutral_Density,layer_index,A) the user supplies the Absolute Salinity SA, conservative Temperature CT and pressure thickness delta_p of many layers, with the first such layer being bounded by the sea surface. The user can request the McDougall-Klocker geostrophic streamfunction on each layer, or he/she can specify the indices of the layers on which the streamfunction is requested in the vector A. For example, A = [6:12] implies that the user wants the streamfunction on layers 6 to 12 inclusive, and the vector Neutral_Density must be consistent with this choice, so that in this case, Neutral_Density must be 7 long. The last argument of the calling sequence to this function must be either 'gn' or 's2' which tells the code whether the interpolating variable for the reference cast is γ^{n} or σ_2.

The two-dimensional gradient of φ^{n} of Eqn. (4) in an approximately neutral surface (ans) is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ in this surface and that at the sea surface, $\mathbf{v}_0$, according to

$$\mathbf{k} \times \nabla_{\mathrm{ans}} \varphi^{\mathrm{n}} = f\mathbf{v} - f\mathbf{v}_0 \qquad \text{or} \qquad \nabla_{\mathrm{ans}} \varphi^{\mathrm{n}} = -\mathbf{k} \times \big(f\mathbf{v} - f\mathbf{v}_0\big). \tag{5}$$

Note that in this function, the reference pressure of the reference surface here is the sea surface. This is because with piecewise-constant vertical profiles of Absolute Salinity and Conservative Temperature and with variable pressure thicknesses of the layers, it is not possible to evaluate a geostrophic streamfunction at a fixed pressure.

A remark on the use of the isopycnal geostrophic streamfunction of McDougall and Klocker (2010) in layered ocean models

Section 8 of McDougall and Klocker (2010) discusses the use of the geostrophic streamfunction given by Eqn. (1) in density-coordinate ocean models. Here we expand a little on this discussion.

If the flow in the ocean is geostrophic then the difference between the horizontal velocity at pressure P and that at the sea surface is given by

$$-\mathbf{k} \times \big(f\mathbf{v} - f\mathbf{v}_0\big) = \Big\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\Big\}, \tag{4}$$

where Φ_0 is the geopotential at the sea surface, and the use of upper case P for pressure serves to remind us that it needs to be in Pa in these expressions. From Eqn. (67) of McDougall and Klocker (2010) we see that $\left\{\frac{1}{\rho}\nabla_z P - \nabla\Phi_0\right\}$ can be expressed in terms of the two-dimensional gradient of φ^{n} (of Eqn. (1) above) in any general surface by

$$\left\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\right\} = \nabla_\sigma\left[\varphi^{\mathrm{n}} + \tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\left(P-\tilde{\tilde{P}}\right)^2\right] - \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\nabla_\sigma\tilde{\tilde{\delta}} + \tfrac{1}{2}\tilde{\tilde{\delta}}\nabla_\sigma P. \qquad (5)$$

This equation demonstrates that in this application to a forward ocean model where all terms on the right-hand side of Eqn. (5) are evaluated during a model run, there is no point including the

$$-\tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\left(P-\tilde{\tilde{P}}\right)^2 \qquad (6)$$

term in the calculation of the geostrophic streamfunction (1) since it is subtracted from this streamfunction when it is used in Eqn. (5) in the forward model.

Hence ,if this suggestion were adopted in a layered ocean model, the geostrophic streamfunction would be

$$\varphi = \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\tilde{\tilde{\delta}}\left(S_{\mathrm{A}},\Theta,p\right) - \int_{P_0}^{P}\tilde{\tilde{\delta}}\,dP' \qquad (7)$$

and the desired horizontal pressure gradient term would be calculated in the forward ocean model using the right-hand side of the following equation,

$$\left\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\right\} = \nabla_\sigma\varphi - \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\nabla_\sigma\tilde{\tilde{\delta}} + \tfrac{1}{2}\tilde{\tilde{\delta}}\nabla_\sigma P. \qquad (8)$$

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org See sections 3.27, 3.30, 3.32 and appendix A.30.

Jackett, D. R. and T. J. McDougall, 1997: A neutral density variable for the world's oceans. *Journal of Physical Oceanography*, **27**, 237-263.

Klocker, A., T. J. McDougall and D. R. Jackett, 2009a: A new method for forming approximately neutral surfaces. *Ocean Sci.*, **5**, 155-172. http://www.ocean-sci.net/5/155/2009/os-5-155-2009.html

Klocker, A., T. J. McDougall and D. R. Jackett, 2009b: Corrigendum to "A new method for forming approximately neutral surfaces" published in Ocean Science, 5, 155–172, 2009, *Ocean Sci.*, **5**, 191-191. http://www.ocean-sci.net/5/191/2009/os-5-191-2009.html

McDougall, T. J. and A. Klocker, 2010: An approximate geostrophic streamfunction for use in density surfaces. *Ocean Modelling*, **32**, 105-117.

Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. Ocean Modelling, 90, pp. 29-43. http://dx.doi.org/10.1016/j.ocemod.2015.04.002

Here follows section 3.30 from the TEOS-10 Manual (IOC *et al.* (2010)).

3.30 Geostrophic streamfunction in an approximately neutral surface

In order to evaluate a relatively accurate expression for the geostrophic streamfunction in an approximately neutral surface a suitable reference seawater parcel $\left(\tilde{\tilde{S}}_{\mathrm{A}}, \tilde{\tilde{\Theta}}, \tilde{\tilde{p}}\right)$ is selected from the approximately neutral surface that one is considering, and the specific volume anomaly $\tilde{\tilde{\delta}}$ is defined as in (3.7.3) above. The approximate geostrophic streamfunction φ^{n} is given by (from McDougall and Klocker (2010))

$$\varphi^{\mathrm{n}} = \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\tilde{\tilde{\delta}}\left(S_{\mathrm{A}},\Theta,p\right) - \tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\left(P-\tilde{\tilde{P}}\right)^{2} - \int_{P_0}^{P}\tilde{\tilde{\delta}}\,dP' . \qquad (3.30.1)$$

This expression is more accurate than the Montgomery and Cunningham geostrophic streamfunctions when used in potential density surfaces, in the ω-surfaces of Klocker *et al.* (2009a,b) and in the Neutral Density surfaces of Jackett and McDougall (1997). That is, in these surfaces $\nabla_n\varphi^{\mathrm{n}} \approx \frac{1}{\rho}\nabla_z P - \nabla\Phi_0 = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right)$ to a very good approximation. In Eqn. (3.30.1) $\rho^{-1}T_{\mathrm{b}}^{\Theta}$ is taken to be the constant value $2.7x10^{-15}\,\mathrm{K}^{-1}(\mathrm{Pa})^{-2}\mathrm{m}^2\mathrm{s}^{-2}$. This approximate isopycnal geostrophic streamfunction of McDougall and Klocker (2010) is available as the function **gsw_geo_strf_isopycnal** in the GSW Toolbox. When the last argument of this function, p_ref, is other than zero, the function returns the isopycnal geostrophic streamfunction with respect to a (deep) reference sea pressure p_ref, rather than with respect to the sea surface at $p = 0$ dbar (i.e. $P = P_0$) as in Eqn. (3.30.1).