Notes on the function gsw_IPV_vs_fNsquared_ratio(SA, CT, p)

This function **gsw_IPV_vs_fNsquared_ratio**(SA,CT,p) evaluates the ratio of the planetary isopycnal-potential-vorticity, IPV, to fN^2, using the 75-term polynomial expression for specific volume is discussed in Roquert *et al.* (2015) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al.* (2010)). For dynamical oceanography we may take the 75-term polynomial function expression for specific volume as essentially reflecting the full accuracy of TEOS-10.

This function **gsw_IPV_vs_fNsquared_ratio**(SA,CT,p) evaluates the expression in Eqn. (3.20.5), namely

$$\frac{IPV}{fN^2} \equiv \frac{-g\rho^{-1}\rho_z^{\Theta}}{N^2} = \frac{\beta^{\Theta}\left(p_{\mathrm{r}}\right)}{\beta^{\Theta}\left(p\right)}\frac{\left[R_\rho/r-1\right]}{\left[R_\rho-1\right]}\,, \tag{3.20.5}$$

where the stability ratio R_ρ is

$$R_\rho = \frac{\alpha^{\Theta}\Theta_z}{\beta^{\Theta}\left(S_{\mathrm{A}}\right)_z}\,, \tag{3.15.1}$$

and the isopycnal slope ratio r (obtainable from **gsw_isopycnal_slope_ratio**) is given by

$$r = \frac{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)/\beta^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)}{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p_{\mathrm{r}}\right)/\beta^{\Theta}\left(S_{\mathrm{A}},\Theta,p_{\mathrm{r}}\right)}\,. \tag{3.17.2}$$

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Roquet, F., G. Madec, T. J. McDougall and P. M. Barker, 2015: Accurate polynomial expressions for the density and specific volume of seawater using the TEOS-10 standard. *Ocean Modelling*, **90**, pp. 29-43. http://dx.doi.org/10.1016/j.ocemod.2015.04.002

Here follows section 3.20 of the TEOS-10 Manual (IOC *et al*. (2010)).

3.20 Potential vorticity

Planetary potential vorticity is the Coriolis parameter f times the vertical gradient of a suitable variable. Potential density is sometimes used for that variable but using potential density (i) involves an inaccurate separation between lateral and diapycnal advection because potential density surfaces are not a good approximation to neutral tangent planes and (ii) incurs the non-conservative baroclinic production term of Eqn. (3.13.4). Using approximately neutral surfaces, "ans", (such as Neutral Density surfaces) provides an optimal separation between the effects of lateral and diapycnal mixing in the potential vorticity equation. In this case the potential vorticity variable is proportional to the reciprocal of the thickness between a pair of closely spaced approximately neutral surfaces.

The evolution equation for planetary potential vorticity is derived by first taking the epineutral "divergence" $\nabla_n \cdot$ of the geostrophic relationship from Eqn. (3.12.1), namely $f\mathbf{v} = g\mathbf{k}\times\nabla_p z$. The projected "divergences" of a two-dimensional vector $\mathbf{a}$ in the neutral tangent plane and in an isobaric surface, are $\nabla_n \cdot \mathbf{a} = \nabla_z \cdot \mathbf{a} + \mathbf{a}_z \cdot \nabla_n z$ and $\nabla_p \cdot \mathbf{a} = \nabla_z \cdot \mathbf{a} + \mathbf{a}_z \cdot \nabla_p z$ from which we find (using Eqn. (3.12.6), $\nabla_n z - \nabla_p z = \nabla_n P / P_z$)

$$\nabla_n \cdot \mathbf{a} = \nabla_p \cdot \mathbf{a} + \mathbf{a}_z \cdot \nabla_n P / P_z . \qquad (3.20.1)$$

Applying this relationship to the two-dimensional vector $f\mathbf{v} = g\mathbf{k}\times\nabla_p z$ we have

$$\nabla_n \cdot \left(f\mathbf{v} \right) = g\,\nabla_p \cdot \left(\mathbf{k}\times\nabla_p z \right) + f\mathbf{v}_z \cdot \nabla_n P / P_z = 0. \qquad (3.20.2)$$

The first part of this expression can be seen to be zero by simply calculating its components, and the second part is zero because the thermal wind vector $\mathbf{v}_z$ points in the direction $\mathbf{k}\times\nabla_n P$ (see Eqn. (3.12.3)). It can be shown that $\nabla_r \cdot \left(f\mathbf{v} \right) = 0$ in any surface r which contains the line $\nabla P \times \nabla \rho$.

Eqn. (3.20.2), namely $\nabla_n \cdot \left(f\mathbf{v} \right) = 0$, can be interpreted as the divergence form of the evolution equation of planetary potential vorticity since

$$\nabla_n \cdot \left(f\mathbf{v} \right) = \nabla_n \cdot \left(\frac{q\mathbf{v}}{\gamma_z} \right) = 0 , \qquad (3.20.3)$$

where $q = f\gamma_z$ is the planetary potential vorticity, being the Coriolis parameter times the vertical gradient of Neutral Density. This instantaneous equation can be averaged in a thickness-weighted sense in density coordinates yielding

$$\nabla_n \cdot \left(\frac{\hat{q}\hat{\mathbf{v}}}{\tilde{\gamma}_z} \right) = -\nabla_n \cdot \overline{\left(\frac{\mathbf{v}''q''}{\gamma_z} \right)} = \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q} \right) , \qquad (3.20.4)$$

where the double-primed quantities are deviations of the instantaneous values from the thickness-weighted mean quantities. Here the epineutral eddy flux of planetary potential vorticity per unit area has been taken to be down the epineutral gradient of $\hat{q}$ with the epineutral diffusivity K. The thickness-weighted mean planetary potential vorticity is

$$\hat{q} \equiv \tilde{\gamma}_z \overline{\left(\frac{q}{\gamma_z} \right)\Bigg|_{\gamma}} = f\tilde{\gamma}_z , \qquad (3.20.5)$$

and the averaging in the above equations is consistent with the difference between the thickness-weighted mean velocity and the velocity averaged *on* the Neutral Density surface, $\hat{\mathbf{v}} - \tilde{\mathbf{v}}$ (the bolus velocity), being $\hat{\mathbf{v}} - \tilde{\mathbf{v}} = K\nabla_n \ln\left(\hat{q}\right)$, since Eqn. (3.20.4) can be written as $\nabla_n \cdot \left(f\hat{\mathbf{v}} \right) = \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q} \right)$ while the average of Eqn. (3.20.3) is $\nabla_n \cdot \left(f\tilde{\mathbf{v}} \right) = 0$.

The divergence form of the mean planetary potential vorticity evolution equation, Eqn. (3.20.4), is quite different to that of a normal conservative variable such as Absolute Salinity or Conservative Temperature in that (i) neither the vertical diffusivity nor the dianeutral velocity makes an appearance, and (ii) there is no temporal tendency term in the equation.

The mean planetary potential vorticity equation (3.20.4) may be put into the advective form by subtracting $\hat{q}$ times the mean continuity equation,

$$\left(\frac{1}{\tilde{\gamma}_z}\bigg|_n \right)_t + \nabla_n \cdot \left(\frac{\hat{\mathbf{v}}}{\tilde{\gamma}_z} \right) + \frac{\tilde{e}_z}{\tilde{\gamma}_z} = 0 , \qquad (3.20.6)$$

from Eqn. (3.20.4), yielding

$$\hat{q}_t\big|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{q} = \tilde{\gamma}_z \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q} \right) + \hat{q}\tilde{e}_z , \qquad (3.20.7)$$

or

$$\hat{q}_t\big|_n + \hat{\mathbf{v}} \cdot \nabla_n \hat{q} + \tilde{e}\hat{q}_z = \frac{\mathrm{d}\hat{q}}{\mathrm{d}t} = \tilde{\gamma}_z \nabla_n \cdot \left(\tilde{\gamma}_z^{-1} K \nabla_n \hat{q} \right) + \left(\hat{q}\tilde{e} \right)_z . \qquad (3.20.8)$$

In this form, it is clear that potential vorticity behaves like a conservative variable as far as epineutral mixing is concerned, but it is quite unlike a normal conservative variable as far as vertical mixing is concerned.

If $\hat{q}$ were a normal conservative variable the last term in Eqn. (3.20.8) would be $\left(D\hat{q}_z\right)_z$ where D is the vertical diffusivity. The term that actually appears in Eqn. (3.20.8), $\left(\hat{q}\tilde{e}\right)_z$, is different to $\left(D\hat{q}_z\right)_z$ by $\left(\hat{q}\tilde{e} - D\hat{q}_z\right)_z = f\left(\tilde{e}\tilde{\gamma}_z - D\tilde{\gamma}_{zz}\right)_z$. Equation (A.22.4) for the mean dianeutral velocity $\tilde{e}$ can be expressed as $\tilde{e} \approx D_z + D\tilde{\gamma}_{zz}/\tilde{\gamma}_z$ if the following three aspects of the non-linear equation of state are ignored; (1) cabbeling and thermobaricity, (2) the vertical variation of the thermal expansion coefficient and the saline contraction coefficient, and (3) the vertical variation of the integrating factor $b\left(x,y,z\right)$ of Eqns. (3.20.10) - (3.20.15) below. Even when ignoring these three different implications of the nonlinear equation of state, the evolution equations (3.20.7) and (3.20.8) of $\hat{q}$ are unlike normal conservation equations because of the extra term

$$\left(\hat{q}\tilde{e} - D\hat{q}_z\right)_z = f\left(\tilde{e}\tilde{\gamma}_z - D\tilde{\gamma}_{zz}\right)_z \approx f\left(D_z\tilde{\gamma}_z\right)_z = \left(D_z\hat{q}\right)_z \tag{3.20.9}$$

on their right-hand sides. This presence of this additional term can result in "unmixing" of $\hat{q}$ in the vertical. Consider a situation where both $\hat{q}$ and $\hat{\Theta}$ are locally linear functions of $\hat{S}_{\mathrm{A}}$ down a vertical water column, so that the $\hat{S}_{\mathrm{A}} - \hat{q}$ and $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ diagrams are both locally straight lines, exhibiting no curvature. Imposing a large amount of vertical mixing at one height (e. g. a delta function of D) will not change the $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ diagram because of the zero $\hat{S}_{\mathrm{A}} - \hat{\Theta}$ curvature (see the water-mass transformation equation (A.23.1)). However, the additional term $\left(D_z\hat{q}\right)_z$ of Eqn. (3.20.9) means that there will be a change in $\hat{q}$ of $\left(D_z\hat{q}\right)_z = \hat{q}D_{zz} + \hat{q}_z D_z \approx \hat{q}D_{zz}$ along the neural tangent plane (that is, in Eqn. (3.20.7)). This is $\hat{q}$ times a negative anomaly at the central height of the extra vertical diffusion, and is $\hat{q}$ times a positive anomaly on the flanking heights above and below the central height. In this way, a delta function of extra vertical diffusion induces structure in the initially straight $\hat{S}_{\mathrm{A}} - \hat{q}$ line which is a telltale sign of "unmixing".

This planetary potential vorticity variable, $\hat{q} = f\tilde{\gamma}_z$, is often mapped on Neutral Density surfaces to give insight into the mean circulation of the ocean on density surfaces. The reasoning is that if the influence of dianeutral advection (the last term in Eqn. (3.20.7)) is small, and the epineutral mixing of $\hat{q}$ is also small, then in a steady ocean $\hat{\mathbf{v}} \cdot \nabla_n \hat{q} = 0$ and the thickness-weighted mean flow on density surfaces $\hat{\mathbf{v}}$ will be along contours of thickness-weighted planetary potential vorticity $\hat{q} = f\tilde{\gamma}_z$.

Because the square of the buoyancy frequency, N^2, accurately represents the vertical static stability of a water column, there is a strong urge to regard fN^2 as the appropriate planetary potential vorticity variable, and to map its contours on Neutral Density surfaces. This urge must be resisted, as spatial maps of fN^2 are significantly different to those of $\hat{q} = f\tilde{\gamma}_z$. To see why this is the case the relationship between the epineutral gradients of $\hat{q}$ and fN^2 will be derived.

For the present purposes Neutral Helicity will be assumed sufficiently small that the existence of neutral surfaces is a good approximation, and we seek the integrating factor $b = b\left(x,y,z\right)$ which allows the construction of Neutral Density surfaces (γ surfaces) according to

$$\frac{\nabla\gamma}{\gamma} = b\left(\beta^{\Theta}\nabla S_{\mathrm{A}} - \alpha^{\Theta}\nabla\Theta\right) = b\left(\frac{\nabla\rho}{\rho} - \kappa\nabla P\right). \tag{3.20.10}$$

Taking the curl of this equation gives

$$\frac{\nabla b}{b} \times \left(\kappa\nabla P - \frac{\nabla\rho}{\rho}\right) = -\nabla\kappa \times \nabla P. \tag{3.20.11}$$

The bracket on the left-hand side is normal to the neutral tangent plane and points in the direction $\mathbf{n} = -\nabla_n z + \mathbf{k}$ and is $g^{-1}N^2\left(-\nabla_n z + \mathbf{k}\right)$. Taking the component of Eqn. (3.20.11) in the direction of the normal to the neutral tangent plane, $\mathbf{n}$, we find

$$\begin{aligned} 0 &= \nabla\kappa\times\nabla P\cdot\mathbf{n} \;=\; \left(\nabla_n\kappa + \kappa_z\mathbf{n}\right)\times\left(\nabla_n P + P_z\mathbf{n}\right)\cdot\mathbf{n} \\ &= \nabla_n\kappa\times\nabla_n P\cdot\mathbf{n} \;=\; \nabla_n\kappa\times\nabla_n P\cdot\mathbf{k} \;=\; \left(\kappa_{S_\mathrm{A}}\nabla_n S_\mathrm{A} + \kappa_\Theta\nabla_n\Theta\right)\times\nabla_n P\cdot\mathbf{k} \\ &= T_\mathrm{b}^\Theta\nabla_n P\times\nabla_n\Theta\cdot\mathbf{k} \;=\; g\,N^{-2}H^\mathrm{n}, \end{aligned} \tag{3.20.12}$$

which simply says that the neutral helicity H^n must be zero in order for the dianeutral component of Eqn. (3.20.11) to hold, that is, $\nabla_n P\times\nabla_n\Theta\cdot\mathbf{k}$ must be zero. Here the equalities $\kappa_{S_\mathrm{A}} = \beta_P^\Theta$ and $\kappa_\Theta = -\,\alpha_P^\Theta$ have been used.

Noting that the ∇b can be written as $\nabla b = \nabla_n b + b_z\mathbf{n}$, Eqn. (3.20.11) becomes

$$g^{-1}N^2\,\nabla_n\ln b\times\left(-\nabla_n z + \mathbf{k}\right) \;=\; -\,P_z\,\nabla_p\kappa\times\left(-\nabla_p z + \mathbf{k}\right), \tag{3.20.13}$$

where $\nabla P = P_z\left(-\nabla_p z + \mathbf{k}\right)$ has been used on the right-hand side, $\left(-\nabla_p z + \mathbf{k}\right)$ being the normal to the isobaric surface. Concentrating on the horizontal components of this equation, $g^{-1}N^2\,\nabla_n\ln b \;=\; -\,P_z\,\nabla_p\kappa$, and using the hydrostatic equation $P_z = -\,g\rho$ gives

$$\nabla_n\ln b \;=\; \rho g^2N^{-2}\,\nabla_p\kappa \;=\; -\,\rho g^2N^{-2}\left(\alpha_P^\Theta\nabla_p\Theta - \beta_P^\Theta\nabla_p S_\mathrm{A}\right). \tag{3.20.14}$$

The integrating factor b defined by Eqn. (3.20.10), that is $b \equiv \left(\rho^l/\gamma\right)\nabla\gamma\cdot\nabla\rho^l\big/(\nabla\rho^l\cdot\nabla\rho^l)$ where $\nabla\rho^l \equiv \rho^l(\beta^\Theta\nabla S_\mathrm{A} - \alpha^\Theta\nabla\Theta)$, allows spatial integrals of $b\,(\beta^\Theta\nabla S_\mathrm{A} - \alpha^\Theta\nabla\Theta) = b\,\nabla\ln\rho^l$ $\approx \nabla\ln\gamma$ to be approximately independent of path for "vertical paths", that is, for paths in surfaces whose normal has zero vertical component.

By analogy with fN^2, the Neutral Surface Potential Vorticity ($NSPV$) is defined as $-g\gamma^{-1}$ times $\hat{q} = f\tilde{\gamma}_z$, so that $NSPV = b\,fN^2$ (having used the vertical component of Eqn. (3.20.9)), so that the ratio of $NSPV$ to fN^2 is found from Eqn. (3.20.14) to be

$$\begin{aligned} \frac{NSPV}{fN^2} \;=\; b \;=\; \frac{\rho^l}{\gamma}\frac{\gamma_z}{\rho_z^l} &= \exp\left\{-\int_\mathrm{ans}\rho g^2\,N^{-2}\left(\alpha_P^\Theta\nabla_p\Theta - \beta_P^\Theta\nabla_p S_\mathrm{A}\right)\cdot d\boldsymbol{l}\right\} \\ &= \exp\left\{\int_\mathrm{ans}\rho g^2\,N^{-2}\nabla_p\kappa\cdot d\boldsymbol{l}\right\}. \end{aligned} \tag{3.20.15}$$

The integral here is taken along an approximately neutral surface (such a Neutral Density surface) from a location where $NSPV$ is equal to fN^2.

The deficiencies of fN^2 as a form of planetary potential vorticity have not been widely appreciated. Even in a lake, the use of fN^2 as planetary potential vorticity is inaccurate since the right-hand side of (3.20.14) is then

$$-\,\rho g^2N^{-2}\,\alpha_P^\Theta\nabla_p\Theta \;=\; \rho g^2N^{-2}\,\alpha_P^\Theta\Theta_z\,\nabla_\Theta P\big/P_z \;=\; -\,\frac{\alpha_P^\Theta}{\alpha^\Theta}\,\nabla_\Theta P, \tag{3.20.16}$$

where the geometrical relationship $\nabla_p\Theta = -\,\Theta_z\,\nabla_\Theta P\big/P_z$ has been used along with the hydrostatic equation. The mere fact that the Conservative Temperature surfaces in a lake have a slope (i. e. $\nabla_\Theta P \neq \mathbf{0}$) means that the spatial variation of contours of fN^2 will not be the same as that of the contours of $NSPV$ in a lake.

In the situation where there is no gradient of Conservative Temperature along a Neutral Density surface ($\nabla_\gamma\Theta = \mathbf{0}$) the contours of $NSPV$ along the Neutral Density surface coincide with those of isopycnal-potential-vorticity (IPV), the potential vorticity defined with respect to the vertical gradient of potential density by $IPV = -\,fg\rho^{-1}\rho_z^\Theta$. IPV is related to fN^2 by (McDougall (1988))

$$\frac{IPV}{fN^2} \;\equiv\; \frac{-\,g\rho^{-1}\rho_z^\Theta}{N^2} \;=\; \frac{\beta^\Theta\left(p_\mathrm{r}\right)}{\beta^\Theta\left(p\right)}\frac{\left[R_\rho/r - 1\right]}{\left[R_\rho - 1\right]} \;=\; \frac{\beta^\Theta\left(p_\mathrm{r}\right)}{\beta^\Theta\left(p\right)}\frac{1}{G^\Theta} \;\approx\; \frac{1}{G^\Theta}\,, \tag{3.20.17}$$

so that the ratio of $NSPV$ to IPV plotted on an approximately neutral surface is given by

$$\frac{NSPV}{IPV} \;=\; \frac{\beta^{\Theta}\left(p\right)}{\beta^{\Theta}\left(p_{\mathrm{r}}\right)}\,\frac{\left[R_{\rho}-1\right]}{\left[R_{\rho}/r-1\right]}\,\exp\left\{\int_{\mathrm{ans}}\rho g^{2}\,N^{-2}\nabla_{p}\kappa\cdot d\boldsymbol{l}\right\}. \tag{3.20.18}$$

You and McDougall (1991) show that because of the highly differentiated nature of potential vorticity, isolines of IPV and $NSPV$ do not coincide even at the reference pressure p_{r} of the potential density variable (see equations (14) – (16) and Figure 14 of that paper). $NSPV$, fN^2 and IPV have the units s^{-3}. The ratio IPV/fN^2, evaluated according to the middle expression in Eqn. (3.20.17), is available in the GSW Oceanographic Toolbox as the function **gsw_IPV_vs_fNsquared_ratio**.