Notes on the GSW code gsw_z_from_p for calculating height *z* from pressure *p*

Height z is measured positive upwards, so it is negative in the ocean. First, note that we use the following version of specific volume anomaly,

$$\delta = \hat{v}\left(S_{\mathrm{A}}, \Theta, p\right) - \hat{v}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p\right). \tag{1}$$

That is, the reference Absolute salinity is the Absolute Salinity of the Standard Ocean, $S_{\mathrm{SO}} \equiv 35.165\,04\ \mathrm{g\ km}^{-1}$, and the reference "temperature" is a fixed value of Conservative Temperature of zero degrees Celsius. Dynamic height anomaly is then defined by Eqn. (3.27.1) of IOC *et al*. (2010) as follows

$$\Psi = -\int_{P_0}^{P} \delta\left(p'\right) dP', \tag{2}$$

where $P_0 = 101\,325\,\mathrm{Pa}$ is the standard atmosphere pressure.

The vertical integral of the hydrostatic equation ($P_z = -g\,\rho$) is

$$\begin{aligned} \int_0^z g\left(z'\right) dz &= -\int_{P_0}^{P} v\left(p'\right) dP' = -\int_{P_0}^{P} \hat{v}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p'\right) dP' + \Psi \\ &= -\hat{h}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p\right) + \Psi . \end{aligned} \tag{3}$$

We use the 25-term based expression (Eqn. A.30.6) for enthalpy, recognizing that because $\Theta = 0°\mathrm{C}$ many of the coefficients on pages 121-122 of the TEOS-10 Manual are zero, so the evaluation of Eqn. (A.30.6) is less computationally expensive than it may appear. Let us write the gravitational acceleration of Eqn. (D.3) of IOC *et al*. (2010) as

$$g = g\left(\phi, 0\right)\left(1 - \gamma\, z\right), \tag{4}$$

so that Eqn. (3) becomes

$$\boxed{\hat{h}^{25}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p\right) + g\left(\phi, 0\right)\left(z - \tfrac{1}{2}\gamma\, z^2\right) = \Psi}\,. \tag{5}$$

In the gsw_z_from_p code we ignore Ψ and solve this quadratic expression for the height z (using the standard quadratic solution equation, but for z^{-1}.) Note again that height z is negative in the ocean.

Notes on the GSW code gsw_p_from_z for calculating pressure *p* from height *z*

In the gsw_p_from_z code we evaluate pressure p using a modified Newton-Raphson iteration procedure so that the pressure so obtained is exactly consistent with the "forward" conversion of z from p via the function gsw_z_from_p.

A good starting point is the Saunders (1981) quadratic (note that the Saunders paper and the corresponding SeaWater code are for depth not height). So, given z we have a zeroth estimate of pressure, p_0, from the Saunders (1981) quadratic expression. Now we want to solve

$$f\left(p\right) = 0, \quad \text{where} \quad f\left(p\right) = \hat{h}^{25}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p\right) + g\left(\phi, 0\right)\left(z - \tfrac{1}{2}\gamma\, z^2\right). \tag{6}$$

The derivative of $f\left(p\right)$ is

$$f'\left(p\right) = 10^4\, \hat{v}^{25}\left(S_{\mathrm{SO}}, 0°\mathrm{C}, p\right), \tag{7}$$

and this is available from appendix K of the TEOS-10 Manual (and since $\Theta = 0°\text{C}$ $\hat{v}^{25}\left(S_{\text{SO}},0°\text{C},p\right)$ is particularly simple to evaluate, having only 10 terms, not 25). The factor of 10^4 in Eqn. (7) is because we want to solve for pressure in dbar rather than in the natural SI unit for pressure of Pa. That is, Eqn. (7) is the derivative of $f\left(p\right)$ with respect to pressure p in dbar.

Calculating $f\left(p\right)$ is computationally expensive, but calculating $f'\left(p\right)$ is cheap, so after finding p_0 we then evaluate $f\left(p_0\right) = \hat{h}^{25}\left(S_{\text{SO}},0°\text{C},p_0\right) + g\left(\phi,0\right)\left(z - \tfrac{1}{2}\gamma z^2\right)$, then calculate $f'\left(p_0\right) = 10^4\,\hat{v}^{25}\left(S_{\text{SO}},0°\text{C},p_0\right)$ and use these values of $f\left(p_0\right)$ and $f'\left(p_0\right)$ to form an intermediate pressure estimate p_1 as

$$p_1 = p_0 - f\left(p_0\right)/f'\left(p_0\right) \ . \tag{8}$$

Then we form $p_{\text{m}} = 0.5\left(p_0 + p_1\right)$ and evaluate $f'\left(p_{\text{m}}\right) = 10^4\,\hat{v}^{25}\left(S_{\text{SO}},0°\text{C},p_{\text{m}}\right)$ and use $f\left(p_0\right)$ and $f'\left(p_{\text{m}}\right)$ to get p_2 from

$$p_2 = p_0 - f\left(p_0\right)/f'\left(p_{\text{m}}\right) \ . \tag{9}$$

This is one full step of the "modified Newton-Raphson" iteration procedure and this one modified step gives pressure to better than $1.6x10^{-10}$ dbar (which is essentially machine precision) down to a height z of -8000m. The gsw_p_from_z function performs this one full iteration of the modified Newton-Raphson iteration.