3.8 Thermobaric coefficient

The thermobaric coefficient quantifies the rate of variation with pressure of the ratio of the thermal expansion coefficient and the saline contraction coefficient. With respect to potential temperature θ the thermobaric coefficient is (McDougall (1987b))

$$T_{\mathrm{b}}^{\theta} = T_{\mathrm{b}}^{\theta}\left(S_{\mathrm{A}},t\,p\right) = \beta^{\theta}\left.\frac{\partial\left(\alpha^{\theta}/\beta^{\theta}\right)}{\partial P}\right|_{S_{\mathrm{A}},\theta} = \left.\frac{\partial\alpha^{\theta}}{\partial P}\right|_{S_{\mathrm{A}},\theta} - \frac{\alpha^{\theta}}{\beta^{\theta}}\left.\frac{\partial\beta^{\theta}}{\partial P}\right|_{S_{\mathrm{A}},\theta}. \tag{3.8.1}$$

This expression for the thermobaric coefficient is most readily evaluated by differentiating an expression for density expressed as a function of potential temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \tilde{\rho}\left(S_{\mathrm{A}},\theta,p\right)$.

With respect to Conservative Temperature Θ the thermobaric coefficient is

$$T_{\mathrm{b}}^{\Theta} = T_{\mathrm{b}}^{\Theta}\left(S_{\mathrm{A}},t\,p\right) = \beta^{\Theta}\left.\frac{\partial\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\partial P}\right|_{S_{\mathrm{A}},\Theta} = \left.\frac{\partial\alpha^{\Theta}}{\partial P}\right|_{S_{\mathrm{A}},\Theta} - \frac{\alpha^{\Theta}}{\beta^{\Theta}}\left.\frac{\partial\beta^{\Theta}}{\partial P}\right|_{S_{\mathrm{A}},\Theta}. \tag{3.8.2}$$

This expression for the thermobaric coefficient is most readily evaluated by differentiating an expression for density expressed as a function of Conservative Temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \hat{\rho}\left(S_{\mathrm{A}},\Theta,p\right)$.

The thermobaric coefficient enters various quantities to do with the path-dependent nature of neutral trajectories and the ill-defined nature of neutral surfaces (see (3.13.1) – (3.13.7)). The thermobaric dianeutral advection associated with the lateral mixing of heat and salt along neutral tangent planes is given by $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\theta}\nabla_{n}\theta\cdot\nabla_{n}P$ or $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_{n}\Theta\cdot\nabla_{n}P$ where $\nabla_{n}\theta$ and $\nabla_{n}\Theta$ are the two-dimensional gradients of either potential temperature or Conservative Temperature along the neutral tangent plane, $\nabla_{n}P$ is the corresponding epineutral gradient of absolute pressure and K is the epineutral diffusion coefficient. Note that the thermobaric dianeutral advection is proportional to the mesoscale eddy flux of "heat" along the neutral tangent plane, $-c_{p}^{0}K\,\nabla_{n}\Theta$, and is independent of the amount of small-scale (dianeutral) turbulent mixing and hence is also independent of the dissipation of mechanical energy ε (Klocker and McDougall (2010a)). It is shown in appendix A.14 below that while the epineutral diffusive fluxes $-K\,\nabla_{n}\theta$ and $-K\,\nabla_{n}\Theta$ are different, the product of these fluxes with their respective thermobaric coefficients is the same, that is, $T_{\mathrm{b}}^{\theta}\nabla_{n}\theta = T_{\mathrm{b}}^{\Theta}\nabla_{n}\Theta$. Hence the thermobaric dianeutral advection e^{Tb} is the same whether it is calculated as $-gN^{-2}K\,T_{\mathrm{b}}^{\theta}\nabla_{n}\theta\cdot\nabla_{n}P$ or as $-gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_{n}\Theta\cdot\nabla_{n}P$. Expressions for T_{b}^{θ} and T_{b}^{Θ} in terms of enthalpy in the functional forms $\tilde{h}\left(S_{\mathrm{A}},\theta,p\right)$ and $\hat{h}\left(S_{\mathrm{A}},\Theta,p\right)$ can be found in appendix P.

Interestingly, for given magnitudes of the epineutral gradients of pressure and Conservative Temperature, the dianeutral advection, $e^{\mathrm{Tb}} = -gN^{-2}K\,T_{\mathrm{b}}^{\Theta}\nabla_{n}\Theta\cdot\nabla_{n}P$, of thermobaricity is maximized when these gradients are parallel, while neutral helicity is maximized when these gradients are perpendicular, since neutral helicity is proportional to $T_{\mathrm{b}}^{\Theta}\left(\nabla_{n}P\times\nabla_{n}\Theta\right)\cdot\mathbf{k}$ (see Eqn. (3.13.2)).

This thermobaric vertical advection process, e^{Tb}, is absent from standard layered ocean models in which the vertical coordinate is a function only of S_{A} and Θ (such as σ_{2}, potential density referenced to 2000 dbar). As described in appendix A.27 below, the isopycnal diffusion of heat and salt in these layered models, caused by both parameterized diffusion along the coordinate and by eddy-resolved motions, does give rise to the cabbeling advection through the coordinate surfaces but does not allow the thermobaric velocity e^{Tb} through these surfaces (Klocker and McDougall (2010a)).

In both the SIA and GSW computer software libraries the thermobaric parameter is output in units of $\mathrm{K}^{-1}\ \mathrm{Pa}^{-1}$.

A.14 Advective and diffusive "heat" fluxes

In section 3.23 and appendices A.8 and A.13 the First Law of Thermodynamics is shown to be practically equivalent to the conservation equation (A.21.15) for Conservative Temperature Θ. We have emphasized that this means that the advection of "heat" is very accurately given as the advection of $c_p^0\Theta$. In this way $c_p^0\Theta$ can be regarded as the "heat content" per unit mass of seawater and the error involved with making this association is approximately 1% of the error in assuming that either $c_p^0\theta$ or $c_p\left(S_\mathrm{A},\theta,0\,\mathrm{dbar}\right)\theta$ is the "heat content" per unit mass of seawater (see also appendix A.21 for a discussion of this point).

The conservative form (A.21.15) implies that the turbulent diffusive flux of heat should be directed down the mean gradient of Conservative Temperature rather than down the mean gradient of potential temperature. In this appendix we quantify the difference between these mean temperature gradients.

Consider first the respective temperature gradients along the neutral tangent plane. From Eqn. (3.11.2) we find that

$$\left(\alpha^\theta/\beta^\theta\right)\nabla_n\theta = \nabla_n S_\mathrm{A} = \left(\alpha^\Theta/\beta^\Theta\right)\nabla_n\Theta, \qquad \text{(A.14.1)}$$

so that the epineutral gradients of θ and Θ are related by the ratios of their respective thermal expansion and saline contraction coefficients, namely

$$\nabla_n\theta = \frac{\left(\alpha^\Theta/\beta^\Theta\right)}{\left(\alpha^\theta/\beta^\theta\right)}\nabla_n\Theta. \qquad \text{(A.14.2)}$$

This proportionality factor between the parallel two-dimensional vectors $\nabla_n\theta$ and $\nabla_n\Theta$ is readily calculated and illustrated graphically. Before doing so we note two other equivalent expressions for this proportionality factor.

The epineutral gradients of θ, Θ and S_A are related by (using $\theta = \hat{\theta}\left(S_\mathrm{A},\Theta\right)$)

$$\nabla_n\theta = \hat{\theta}_\Theta\nabla_n\Theta + \hat{\theta}_{S_\mathrm{A}}\nabla_n S_\mathrm{A}, \qquad \text{(A.14.3)}$$

and using the neutral relationship $\nabla_n S_\mathrm{A} = \left(\alpha^\Theta/\beta^\Theta\right)\nabla_n\Theta$ we find

$$\nabla_n\theta = \left(\hat{\theta}_\Theta + \left[\alpha^\Theta/\beta^\Theta\right]\hat{\theta}_{S_\mathrm{A}}\right)\nabla_n\Theta. \qquad \text{(A.14.4)}$$

Also, in section 3.13 we found that $T_b^\theta\nabla_n\theta = T_b^\Theta\nabla_n\Theta$, so that we can write the equivalent expressions

$$\frac{\left|\nabla_n\theta\right|}{\left|\nabla_n\Theta\right|} = \frac{\left(\alpha^\Theta/\beta^\Theta\right)}{\left(\alpha^\theta/\beta^\theta\right)} = \frac{T_b^\Theta}{T_b^\theta} = \hat{\theta}_\Theta + \left[\alpha^\Theta/\beta^\Theta\right]\hat{\theta}_{S_\mathrm{A}}, \qquad \text{(A.14.5)}$$

and it can be shown that $\alpha^\Theta/\alpha^\theta = \hat{\theta}_\Theta$ and $\beta^\theta/\beta^\Theta = \left(1+\left[\alpha^\Theta/\beta^\Theta\right]\hat{\theta}_{S_\mathrm{A}}/\hat{\theta}_\Theta\right)$, that is, $\beta^\theta = \beta^\Theta + \alpha^\Theta\,\hat{\theta}_{S_\mathrm{A}}/\hat{\theta}_\Theta$. The partial derivatives $\hat{\theta}_\Theta$ and $\hat{\theta}_{S_\mathrm{A}}$ in the last part of Eqn. (A.14.5) are both independent of pressure while $\alpha^\Theta/\beta^\Theta$ is a function of pressure. This ratio, Eqn. (A.14.5), of the epineutral gradients of θ and Θ is shown in Figure A.14.1 at $p = 0$, indicating that the epineutral gradient of potential temperature is sometimes more that 1% different to that of Conservative Temperature. This ratio $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ is only a weak function of pressure. This ratio, $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ (i.e. Eqn. (A.14.5)), is available in the GSW Oceanographic Toolbox as function **gsw_ntp_pt_vs_CT_ratio_CT25**.

Similarly to Eqn. (A.14.3), the vertical gradients are related by

$$\theta_z = \hat{\theta}_\Theta\,\Theta_z + \hat{\theta}_{S_\mathrm{A}}S_{\mathrm{A}_z}, \qquad \text{(A.14.6)}$$

and using the definition, Eqn. (3.15.1), of the stability ratio we find that

$$\frac{\theta_z}{\Theta_z} = \hat{\theta}_\Theta + R_\rho^{-1}\left[\alpha^\Theta/\beta^\Theta\right]\hat{\theta}_{S_\mathrm{A}}. \qquad \text{(A.14.7)}$$

For values of the stability ratio R_ρ close to unity, the ratio θ_z/Θ_z is close to the values of $|\nabla_n\theta|/|\nabla_n\Theta|$ shown in Figure A.14.1. For other values of R_ρ, Eqn. (A.14.7) can be calculated and plotted.

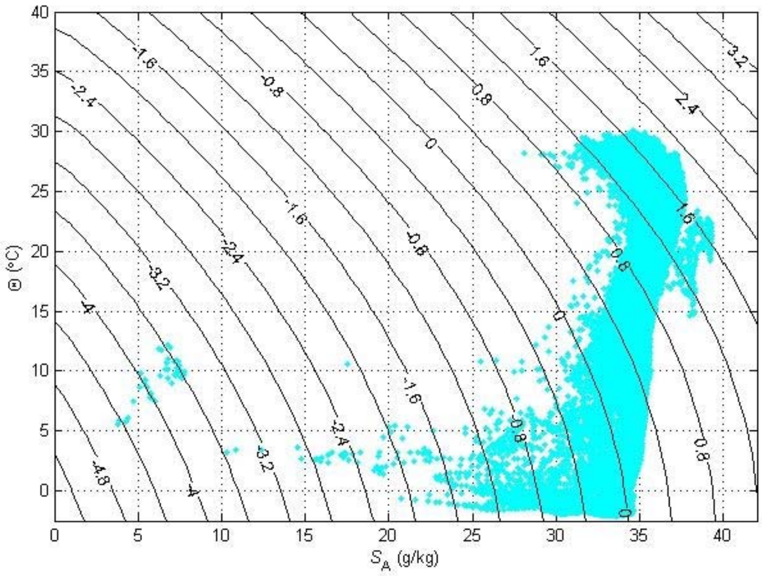

Figure A.14.1. Contours of $\left(|\nabla_n\theta|/|\nabla_n\Theta| \; - \; 1\right) \times 100\%$ at $p = 0$, showing the percentage difference between the epineutral gradients of θ and Θ. The blue dots are from the ocean atlas of Gouretski and Koltermann (2004) at $p = 0$.

As noted in section 3.8 the dianeutral advection of thermobaricity is the same when quantified in terms of potential temperature as when done in terms of Conservative Temperature. The same is not true of the dianeutral velocity caused by cabbeling. The ratio of the cabbeling dianeutral velocity calculated using potential temperature to that using Conservative Temperature is given by $\left(C_\mathrm{b}^\theta \nabla_n\theta \cdot \nabla_n\theta\right)\big/\left(C_\mathrm{b}^\Theta \nabla_n\Theta \cdot \nabla_n\Theta\right)$ (see section 3.9) which can be expressed as

$$\frac{C_\mathrm{b}^\theta\,|\nabla_n\theta|^2}{C_\mathrm{b}^\Theta\,|\nabla_n\Theta|^2} \;=\; \frac{C_\mathrm{b}^\theta}{C_\mathrm{b}^\Theta}\frac{\left(\alpha^\Theta/\beta^\Theta\right)^2}{\left(\alpha^\theta/\beta^\theta\right)^2} \;=\; \frac{C_\mathrm{b}^\theta}{C_\mathrm{b}^\Theta}\left(\frac{T_\mathrm{b}^\Theta}{T_\mathrm{b}^\theta}\right)^2 \;=\; \frac{C_\mathrm{b}^\theta}{C_\mathrm{b}^\Theta}\left(\hat{\theta}_\Theta + \left[\alpha^\Theta/\beta^\Theta\right]\hat{\theta}_{S_\mathrm{A}}\right)^2, \qquad \text{(A.14.8)}$$

and this is contoured in Fig. A.14.2. While the ratio of Eqn. (A.14.8) is not exactly unity, it varies relatively little in the oceanographic range, indicating that the dianeutral advection due to cabbeling estimated using θ or Θ are within half a percent of each other at $p = 0$.

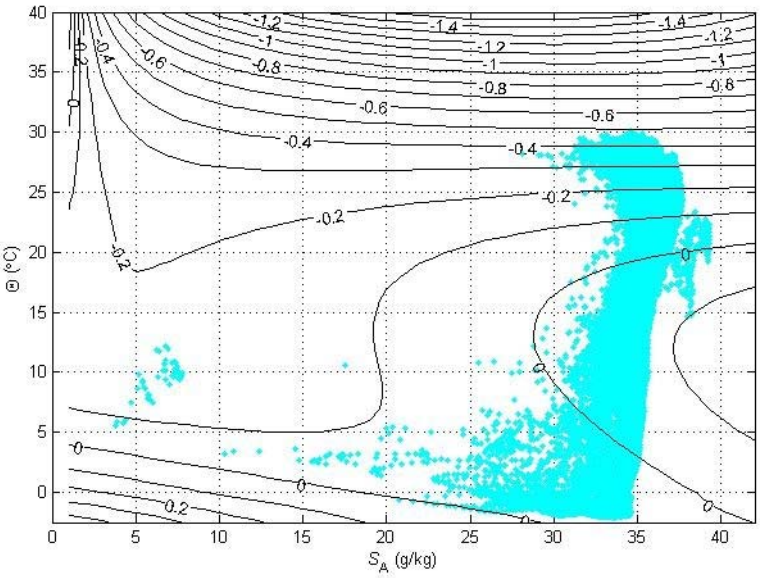

Figure A.14.2. Contours of the percentage difference of $\left(C_{\mathrm{b}}^{\theta}\left|\nabla_{n}\theta\right|^{2}\right)\Big/\left(C_{\mathrm{b}}^{\Theta}\left|\nabla_{n}\Theta\right|^{2}\right)$ from unity at $p = 0$ dbar.