3.20 Potential vorticity

Planetary potential vorticity is the Coriolis parameter f times the vertical gradient of a suitable variable. Potential density is sometimes used for that variable but using potential density (i) involves an inaccurate separation between lateral and diapycnal advection because potential density surfaces are not a good approximation to neutral tangent planes and (ii) incurs the non-conservative baroclinic production term of Eqn. (3.13.4). Using approximately neutral surfaces, "ans", (such as Neutral Density surfaces) provides an optimal separation between the effects of lateral and diapycnal mixing in the potential vorticity equation. In this case the potential vorticity variable is proportional to the reciprocal of the thickness between a pair of closely spaced approximately neutral surfaces. This planetary potential vorticity variable is called Neutral-Surface-Potential-Vorticity ($NSPV$ for short) and is related to fN^2 by

$$NSPV \equiv -g\rho^{-1} f\gamma_z^{\mathrm{n}} \;\approx\; fN^2 \exp\left\{-\int_{\mathrm{ans}} \rho g^2 N^{-2}\, T_{\mathrm{b}}^{\Theta}\left(\nabla_a\Theta - \Theta_P \nabla_a P\right)\cdot d\boldsymbol{l}\right\}. \qquad (3.20.1)$$

The exponential expression was derived by McDougall (1988) (his equation (47)) and is approximate because the variation of the saline contraction coefficient β^{Θ} with pressure was neglected in comparison with the larger proportional change in the thermal expansion coefficient α^{Θ} with pressure. The integral in Eqn. (3.20.1) is taken along an approximately neutral surface from a location where $NSPV$ is equal to fN^2. Interestingly the combination $\nabla_a\Theta - \Theta_P\nabla_a P$ is simply the isobaric gradient of Conservative Temperature, $\nabla_P\Theta$, which is almost the same as the horizontal gradient, $\nabla_z\Theta$. A more accurate version of this equation which does not ignore the variation of the saline contraction coefficient can be shown to be

$$\begin{aligned} NSPV \equiv -g\rho^{-1} f\gamma_z^{\mathrm{n}} &= fN^2 \exp\left\{-\int_{\mathrm{ans}} g^2 N^{-2}\left((\rho\alpha^{\Theta})_P \nabla_P\Theta - (\rho\beta^{\Theta})_P \nabla_P S_{\mathrm{A}}\right)\cdot d\boldsymbol{l}\right\} \\ &= fN^2 \exp\left\{\int_{\mathrm{ans}} g^2 N^{-2}\nabla_P(\rho\kappa)\cdot d\boldsymbol{l}\right\}. \end{aligned} \qquad (3.20.2)$$

The exponential factor in Eqn. (3.20.2) is approximately the integrating factor b, defined as $b \equiv \nabla\gamma^{\mathrm{n}}\cdot\nabla\rho^l / (\nabla\rho^l\cdot\nabla\rho^l)$ where $\nabla\rho^l \equiv \rho(\beta^{\Theta}\nabla S_{\mathrm{A}} - \alpha^{\Theta}\nabla\Theta)$, which allows spatial integrals of $\rho b(\beta^{\Theta}\nabla S_{\mathrm{A}} - \alpha^{\Theta}\nabla\Theta) = b\nabla\rho^l \approx \nabla\gamma^{\mathrm{n}}$ to be approximately independent of path for "vertical paths", that is, for paths in surfaces whose normal has zero vertical component.

The gradient ∇_{a} of fN^2 is related to that of $NSPV$ by (from Eqns. (3.20.2) and (3.20.1))

$$\nabla_a\left(\ln fN^2\right) - \nabla_a\left(\ln NSPV\right) = -g^2 N^{-2}\nabla_P(\rho\kappa) \;\approx\; \rho g^2 N^{-2} T_{\mathrm{b}}^{\Theta}\left(\nabla_a\Theta - \Theta_P\nabla_a P\right). \qquad (3.20.3)$$

The deficiencies of fN^2 as a form of planetary potential vorticity have not been widely appreciated. Even in a lake, and also in the simple situation where temperature does not vary along a density surface ($\nabla_a\Theta = \mathbf{0}$), the use of fN^2 as planetary potential vorticity is inaccurate since the right-hand side of (3.20.3) is then approximately

$$-\rho g^2 N^{-2} T_{\mathrm{b}}^{\Theta}\Theta_P\nabla_a P \;=\; \frac{R_\rho}{\alpha^{\Theta}\left[R_\rho - 1\right]}\, T_{\mathrm{b}}^{\Theta}\nabla_a P, \qquad (3.20.4)$$

and the mere fact that the density surface has a slope (i. e. $\nabla_a P \neq \mathbf{0}$) means that the contours of fN^2 will not be parallel to contours of $NSPV$ on the density surface. (In this situation (where $\nabla_a\Theta = \mathbf{0}$) the contours of $NSPV$ along approximately neutral surfaces coincide with those of isopycnal-potential-vorticity (IPV), the potential vorticity defined with respect to the vertical gradient of potential density by $IPV = -fg\rho^{-1}\rho_z^{\Theta}$).

IPV is related to fN^2 by (McDougall (1988))

$$\frac{IPV}{fN^2} \equiv \frac{-g\rho^{-1}\rho_z^{\Theta}}{N^2} = \frac{\beta^{\Theta}(p_{\mathrm{r}})}{\beta^{\Theta}(p)}\frac{\left[R_\rho/r - 1\right]}{\left[R_\rho - 1\right]} = \frac{\beta^{\Theta}(p_{\mathrm{r}})}{\beta^{\Theta}(p)}\frac{1}{G^{\Theta}} \approx \frac{1}{G^{\Theta}}, \qquad (3.20.5)$$

so that the ratio of $NSPV$ to IPV plotted on an approximately neutral surface is given by

$$\frac{NSPV}{IPV} = \frac{\beta^{\Theta}(p)}{\beta^{\Theta}(p_{\mathrm{r}})}\frac{\left[R_{\rho}-1\right]}{\left[R_{\rho}/r-1\right]}\exp\left\{\int_{\mathrm{ans}} g^{2}\,N^{-2}\nabla_{P}(\rho\kappa)\cdot d\boldsymbol{l}\right\}. \tag{3.20.6}$$

You and McDougall (1991) show that because of the highly differentiated nature of potential vorticity, isolines of IPV and $NSPV$ do not coincide even at the reference pressure p_{r} of the potential density variable (see equations (14) – (16) and Figure 14 of that paper). $NSPV$, fN^{2} and IPV have the units s^{-3}. The ratio IPV/fN^{2} is available in the GSW computer software library as the function **gsw_IPV_vs_fNsquared_ratio_CT25**.