A.12 Differential relationships between η, θ, Θ and S_{A}

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\,\mathrm{d}\eta+\mu(p)\,\mathrm{d}S_{\mathrm{A}} &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\,\mathrm{d}\theta \;+\; \big[\mu(p)-(T_0+t)\,\mu_T(0)\big]\mathrm{d}S_{\mathrm{A}} \\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\,\mathrm{d}\Theta \;+\; \left[\mu(p)-\frac{(T_0+t)}{(T_0+\theta)}\mu(0)\right]\mathrm{d}S_{\mathrm{A}}\,.\end{aligned} \qquad \text{(A.12.1)}$$

The quantity $\mu(p)\,\mathrm{d}S_{\mathrm{A}}$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\,\mathrm{d}\eta \;=\; c_p(0)\,\mathrm{d}\theta-(T_0+\theta)\,\mu_T(0)\,\mathrm{d}S_{\mathrm{A}} \;=\; c_p^0\,\mathrm{d}\Theta-\mu(0)\,\mathrm{d}S_{\mathrm{A}}\,. \qquad \text{(A.12.2)}$$

From this follows all the following partial derivatives between η, θ, Θ and S_{A},

$$\Theta_\theta\big|_{S_{\mathrm{A}}} = c_p(S_{\mathrm{A}},\theta,0)/c_p^0, \qquad \Theta_{S_{\mathrm{A}}}\big|_\theta = \big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p^0, \qquad \text{(A.12.3)}$$

$$\Theta_\eta\big|_{S_{\mathrm{A}}} = (T_0+\theta)/c_p^0, \qquad \Theta_{S_{\mathrm{A}}}\big|_\eta = \mu(S_{\mathrm{A}},\theta,0)/c_p^0, \qquad \text{(A.12.4)}$$

$$\theta_\eta\big|_{S_{\mathrm{A}}} = (T_0+\theta)/c_p(S_{\mathrm{A}},\theta,0), \qquad \theta_{S_{\mathrm{A}}}\big|_\eta = (T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)/c_p(S_{\mathrm{A}},\theta,0), \qquad \text{(A.12.5)}$$

$$\theta_\Theta\big|_{S_{\mathrm{A}}} = c_p^0/c_p(S_{\mathrm{A}},\theta,0), \qquad \theta_{S_{\mathrm{A}}}\big|_\Theta = -\big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p(S_{\mathrm{A}},\theta,0), \qquad \text{(A.12.6)}$$

$$\eta_\theta\big|_{S_{\mathrm{A}}} = c_p(S_{\mathrm{A}},\theta,0)/(T_0+\theta), \qquad \eta_{S_{\mathrm{A}}}\big|_\theta = -\mu_T(S_{\mathrm{A}},\theta,0), \qquad \text{(A.12.7)}$$

$$\eta_\Theta\big|_{S_{\mathrm{A}}} = c_p^0/(T_0+\theta), \qquad \eta_{S_{\mathrm{A}}}\big|_\Theta = -\mu(S_{\mathrm{A}},\theta,0)/(T_0+\theta). \qquad \text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}(S_{\mathrm{A}},\Theta)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}(S_{\mathrm{A}},\Theta)$, namely $\hat{\theta}_\Theta$, $\hat{\theta}_{S_{\mathrm{A}}}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_{\mathrm{A}}\Theta}$ and $\hat{\theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_\Theta = \frac{1}{\tilde{\Theta}_\theta}, \quad \hat{\theta}_{S_{\mathrm{A}}} = -\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}, \quad \hat{\theta}_{\Theta\Theta} = -\frac{\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \quad \hat{\theta}_{S_{\mathrm{A}}\Theta} = -\frac{\tilde{\Theta}_{\theta S_{\mathrm{A}}}}{\big(\tilde{\Theta}_\theta\big)^2}+\frac{\tilde{\Theta}_{S_{\mathrm{A}}}\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \qquad \text{(A.12.9a,b,c,d)}$$

$$\text{and} \quad \hat{\theta}_{S_{\mathrm{A}}S_{\mathrm{A}}} = -\frac{\tilde{\Theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}}{\tilde{\Theta}_\theta} + 2\,\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}\frac{\tilde{\Theta}_{\theta S_{\mathrm{A}}}}{\tilde{\Theta}_\theta} - \left(\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}\right)^2\frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_\theta}, \qquad \text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_\theta$, $\tilde{\Theta}_{S_{\mathrm{A}}}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_{\mathrm{A}}}$ and $\tilde{\Theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}(S_{\mathrm{A}},\theta)$ from the TEOS-10 Gibbs function.

… and an excerpt from appendix P

The partial derivatives with respect to Θ and with respect to θ, both at constant S_{A} and p, and the partial derivatives with respect to S_{A}, are related by

$$\left.\frac{\partial}{\partial\Theta}\right|_{S_{\mathrm{A}},p} = \frac{1}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_{\mathrm{A}},p}, \quad \text{and} \quad \left.\frac{\partial}{\partial S_{\mathrm{A}}}\right|_{\Theta,p} = \left.\frac{\partial}{\partial S_{\mathrm{A}}}\right|_{\theta,p} - \frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_{\mathrm{A}},p}. \qquad \text{(P.13a,b)}$$

Use of these expressions, acting on entropy yields (with $p=0$ everywhere, and using Eqn. (P.7) [or Eqn. (A.12.8b)] and Eqn. (P.8))

$$\hat{\eta}_{\Theta} = \frac{\tilde{\eta}_{\theta}}{\tilde{\Theta}_{\theta}} \equiv \frac{c_p^0}{(T_0+\theta)}, \quad \hat{\eta}_{\Theta\Theta} = -\frac{1}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0+\theta)^2}, \quad \hat{\eta}_{S_\mathrm{A}} = -\frac{\tilde{\mu}(S_\mathrm{A}, \theta, 0)}{(T_0+\theta)}, \qquad \text{(P.14a,b,c)}$$

$$\hat{\eta}_{S_\mathrm{A}\Theta} = \frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0+\theta)^2}, \quad \text{and} \quad \hat{\eta}_{S_\mathrm{A}S_\mathrm{A}} = -\frac{\tilde{\mu}_{S_\mathrm{A}}(S_\mathrm{A}, \theta, 0)}{(T_0+\theta)} - \frac{\left(\tilde{\Theta}_{S_\mathrm{A}}\right)^2}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0+\theta)^2}, \qquad \text{(P.15a,b)}$$

in terms of the partial derivatives of the exact polynomial expressions (P.11b) and (P.12).