Notes on the function gsw_enthalpy_diff_CT25

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From appendix A.30 of the TEOS-10 Manual (IOC *et al*. (2010)) we have the following expression for the specific enthalpy of seawater, based on the computationally efficient 25-term expression for specific volume (McDougall *et al*. (2010)),

$$\begin{aligned}\hat{h}^{25}\left(S_{\mathrm{A}},\Theta,p\right) \;=\; & c_p^0\Theta \;+\; 10^4\left(\frac{a_2}{b_2} - \frac{2a_3b_1}{b_2^2}\right)p \;+\; 10^4\,\frac{a_3}{2b_2}\,p^2 \\ & + \frac{M^*}{2b_2}\ln\left(1+\frac{2b_1}{b_0}\,p + \frac{b_2}{b_0}\,p^2\right) \;+\; \frac{N^* - \frac{b_1}{b_2}M^*}{(B-A)}\ln\left(1 \;+\; p\,\frac{b_2}{A}\,\frac{(B-A)}{(B+b_2p)}\right).\end{aligned} \tag{1}$$

This has been written in terms of $A = b_1 - \sqrt{b_1^2 - b_0b_2}$ and $B = b_1 + \sqrt{b_1^2 - b_0b_2}$, and the coefficients a_0, a_1, a_2, a_3 and b_0, b_1, b_2 and M^* and N^* which are all defined in McDougall *et al*. (2010) and in the TEOS-10 Manual. All of these coefficients are functions of only Absolute Salinity and Conservative Temperature; that is, they are independent of pressure.

The function **gsw_enthalpy_diff_CT25** returns the difference between the specific enthalpy of two seawater parcels, both having the same Absolute Salinity and Conservative Temperature, but having different pressures. The two pressures are labeled p^{de} and p^{sh} (for "deep" and "shallow" respectively) and the **gsw_enthalpy_diff_CT25** code returns $\hat{h}^{25}\left(S_{\mathrm{A}},\Theta,p^{\mathrm{de}}\right) - \hat{h}^{25}\left(S_{\mathrm{A}},\Theta,p^{\mathrm{sh}}\right)$ calculated from Eqn. (1) according to

$$\begin{aligned}\hat{h}^{25}\left(S_{\mathrm{A}},\Theta,p^{\mathrm{de}}\right) - \hat{h}^{25}\left(S_{\mathrm{A}},\Theta,p^{\mathrm{sh}}\right) \;=\; & 10^4\left(p^{\mathrm{de}} - p^{\mathrm{sh}}\right)\left[\left(\frac{a_2}{b_2} - \frac{2a_3b_1}{b_2^2}\right) \;+\; \frac{a_3}{2b_2}\left(p^{\mathrm{de}} + p^{\mathrm{sh}}\right)\right] \\ & + \frac{M^*}{2b_2}\ln\left(1 \;+\; \left(p^{\mathrm{de}} - p^{\mathrm{sh}}\right)\frac{\left[2b_1 + b_2\left(p^{\mathrm{de}} + p^{\mathrm{sh}}\right)\right]}{\left[b_0 + p^{\mathrm{sh}}\left(2b_1 + b_2p^{\mathrm{sh}}\right)\right]}\right) \\ & + \frac{N^* - \frac{b_1}{b_2}M^*}{(B-A)}\ln\left(1 \;+\; \left(p^{\mathrm{de}} - p^{\mathrm{sh}}\right)\frac{b_2(B-A)}{\left(B + b_2p^{\mathrm{de}}\right)\left(A + b_2p^{\mathrm{sh}}\right)}\right).\end{aligned} \tag{2}$$

Below, for reference is section 3.32 and appendix A.30 of the TEOS-10 Manual (IOC *et al*. (2010))

3.32 Pressure to height conversion

When vertically integrating the hydrostatic equation $P_z = -g\rho$ in the context of an ocean model where Absolute Salinity S_{A} and Conservative Temperature Θ (or potential temperature θ) are piecewise constant in the vertical, the geopotential (Eqn. (3.24.2))

$$\Phi \;=\; \Phi^0 - \int_{P_0}^{P} v\left(p'\right)\,dP', \tag{3.32.1}$$

can be evaluated as a series of exact differences. If there are a series of layers of index i separated by pressures p^i and p^{i+1} (with $p^{i+1} > p^i$) then the integral can be expressed (making use of (3.7.5), namely $h_P\big|_{S_{\mathrm{A}},\Theta} = \hat{h}_P = v$) as a sum over n layers of the differences in specific enthalpy so that

$$\Phi \;=\; \Phi^0 - \int_{P_0}^{P} v\left(p'\right)\,dP' \;=\; \Phi^0 - \sum_{i=1}^{n-1}\left[\hat{h}\left(S_{\mathrm{A}}^i,\Theta^i,p^{i+1}\right) - \hat{h}\left(S_{\mathrm{A}}^i,\Theta^i,p^i\right)\right]. \tag{3.32.2}$$

A.30 Computationally efficient 25-term expressions for the density of seawater in terms of Θ and θ

Ocean models to date have treated their salinity and temperature variables as being Practical Salinity S_{P} and potential temperature θ. As the full implications of TEOS-10 are incorporated into ocean models they will need to carry Preformed Salinity S_* and Conservative Temperature Θ as conservative variables (as discussed in appendices A.20 and A.21), and a computationally efficient expression for density in terms of Absolute Salinity S_{A} and Conservative Temperature Θ will be needed.

Following the work of McDougall *et al*. (2003) and Jackett *et al*. (2006), the TEOS-10 density ρ has been approximated by a rational function of the same form as in those papers. The fitted expression is the ratio of two polynomials of $(S_{\mathrm{A}}, \Theta, p)$

$$\rho \approx \rho^{25} = P_{\mathrm{num}}^{\rho 25} \big/ P_{\mathrm{denom}}^{\rho 25} . \tag{A.30.1}$$

The density data has been fitted in a "funnel" of data points in (S_{A}, t, p) space which is described in more detail in McDougall *et al.* (2010b). The "funnel" extends to a pressure of $8000\ \mathrm{dbar}$. At the sea surface the "funnel" covers the full range of temperature and salinity while for pressures greater than 5500 dbar, the maximum temperature of the fitted data is $12°\mathrm{C}$ and the minimum Absolute Salinity is $u_{\mathrm{PS}}30\ \mathrm{g\ kg}^{-1}$. That is, the fit has been performed over a region of parameter space which includes water that is approximately $10°\mathrm{C}$ warmer and $5\ \mathrm{g\ kg}^{-1}$ fresher in the deep ocean than exists in the present ocean (see Figure 1 of Jackett *et al*. (2006)). Table K.1 of appendix K contains the 25 coefficients of the expression (A.30.1) for density in terms of $(S_{\mathrm{A}}, \Theta, p)$.

As outlined in appendix K, this 25-term rational-function expression for ρ yields the thermal expansion and haline contraction coefficients, α^{Θ} and β^{Θ}, that are essentially as accurate as those derived from the full TEOS-10 Gibbs function for data in the "oceanographic funnel". The same cannot be claimed for the sound speed derived by differentiating Eqn. (A.30.1) with respect to pressure; this sound speed has an rms error in the "funnel" of almost $0.25\ \mathrm{m\ s}^{-1}$ whereas TEOS-10 fits the available sound speed data with an rms error of only $0.035\ \mathrm{m\ s}^{-1}$.

In dynamical oceanography it is the thermal expansion and haline contraction coefficients α^{Θ} and β^{Θ} which are the most important aspects of the equation of state since the "thermal wind" is proportional to $\alpha^{\Theta}\nabla_p\Theta - \beta^{\Theta}\nabla_p S_{\mathrm{A}}$ and the vertical static stability is given in terms of the buoyancy frequency N by $g^{-1}N^2 = \alpha^{\Theta}\Theta_z - \beta^{\Theta}(S_{\mathrm{A}})_z$. Hence for dynamical oceanography we may take the 25-term rational function expression for density, Eqn. (A.30.1), as essentially reflecting the full accuracy of TEOS-10. This is confirmed in Fig. A.30.1 where the error in using the 25-term expression for density to calculate the isobaric northward density gradient is shown. The vertical axis on this figure is the magnitude of the difference in the northward isobaric density gradient in the world ocean below $1000\,\mathrm{m}$ when evaluated using Eqn. (A.30.1) versus using the full TEOS-10 Gibbs function. The scales of the axes of this figure have been chosen to be the same as those of Fig. A.5.1 of appendix A.5 so that the smallness of the errors associated with using the 25-term density expression can be appreciated. The errors represented in Fig. A.30.1 represent perhaps half of the remaining uncertainty in our knowledge of seawater properties, and by comparing Figs. A.30.1 and A.5.1 it is clear that the much more important issue is to properly represent the effects of seawater composition on seawater density. The rms value of the vertical axis in Fig. A.30.1 is 11.4% of that of Fig. A.5.1.

McDougall *et al*. (2010b) have also provided a 25-term rational-function expression for density in terms of $(S_{\mathrm{A}}, \theta, p)$. The 25 coefficients can be found in Table K.2 of appendix K. As an approximation to density, this 25-term rational function is approximately as accurate as the one described above in terms of $(S_{\mathrm{A}}, \Theta, p)$. It must be emphasized though that an ocean model that treated potential temperature as a conservative variable would make errors in its treatment of heat fluxes, as described in appendices A.13, A.14 and A.17, and as illustrated in Figures A.13.1, A.14.1 and A.17.1.

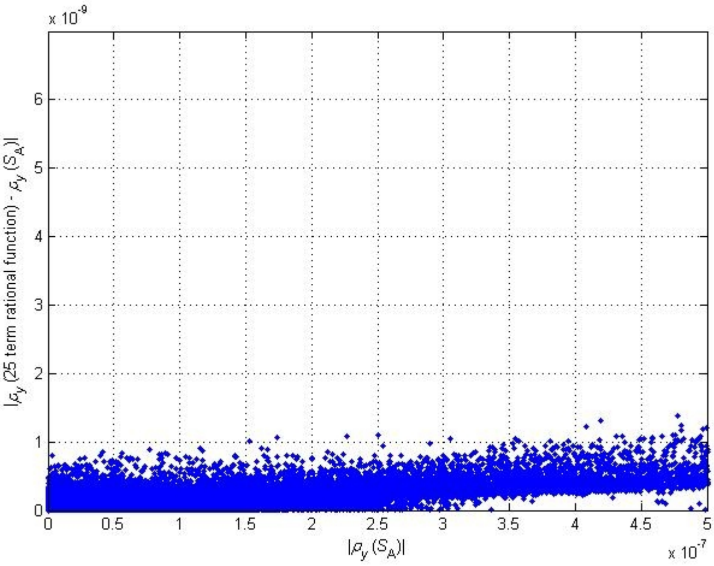

Figure A.30.1. The northward density gradient at constant pressure (the horizontal axis) for data in the world ocean atlas of Gouretski and Koltermann (2004) for $p > 1000$ dbar. The vertical axis is the magnitude of the difference between evaluating the density gradient using the 25-term expression Eqn. (A.30.1) instead of using the full TEOS-10 expression, using Absolute Salinity S_A as the salinity argument in both cases.

Appendix P describes how an expression for the enthalpy of seawater in terms of Conservative Temperature, specifically the functional form $\hat{h}(S_A, \Theta, p)$, together with an expression for entropy in the form $\hat{\eta}(S_A, \Theta)$, can be used as an alternative thermodynamic potential to the Gibbs function $g(S_A, t, p)$. The need for the functional form $\hat{h}(S_A, \Theta, p)$ also arises in section 3.32 and in Eqns. (3.26.3) and (3.29.1). The 25-term expression, Eqn. (A.30.1), for $\rho^{25} = \hat{\rho}^{25}(S_A, \Theta, p)$ can be used to find a closed expression for $\hat{h}(S_A, \Theta, p)$ by integrating the reciprocal of $\hat{\rho}^{25}(S_A, \Theta, p)$ with respect to pressure (in Pa), since $\hat{h}_P = v = \rho^{-1}$ (see Eqn. (2.8.3)).

The 25-term expression for specific volume, Eqn. (A.30.1), is first written explicitly as the ratio of two polynomials in sea pressure p (in dbar) as

$$\hat{v}^{25} = \frac{1}{\hat{\rho}^{25}} = \frac{a_0 + a_1 p + a_2 p^2 + a_3 p^3}{b_0 + 2b_1 p + b_2 p^2}, \qquad \text{(A.30.2)}$$

where the coefficients a_0 to a_3 and b_0 to b_2 are the following functions of S_A and Θ

$$a_0 = 1 + c_{14}\Theta + c_{15}\Theta^2 + c_{16}\Theta^3 + c_{17}\Theta^4 + c_{18}S_A + c_{19}S_A\,\Theta + c_{20}S_A\,\Theta^3 + c_{21}\left(S_A\right)^{1.5} + c_{22}\left(S_A\right)^{1.5}\Theta^2,$$

$$a_1 = c_{23},$$

$$a_2 = c_{24}\Theta^3,$$

$$a_3 = c_{25}\,\Theta,$$

$$b_0 = c_1 + c_2\Theta + c_3\Theta^2 + c_4\Theta^3 + c_5 S_A + c_6 S_A \Theta + c_7\left(S_A\right)^2,$$

$$b_1 = 0.5\left(c_8 + c_9\Theta^2 + c_{10}S_A\right),$$

$$b_2 = c_{11} + c_{12}\Theta^2,$$

and the numbered coefficients c_1 to c_{25} are so identified in Table K.1 (note that $c_{13} = 1$).

It is not difficult to rearrange Eqn. (A.30.2) into the form

$$\hat{v}^{25} = \hat{v}^{25}(S_A,\Theta,p) = \left(\frac{a_2}{b_2} - \frac{2a_3b_1}{b_2^2}\right) + \frac{a_3}{b_2}p + \frac{N + Mp}{b_0 + 2b_1p + b_2p^2}, \qquad \text{(A.30.3)}$$

where N and M are given by

$$N = a_0 + \frac{2a_3b_0b_1}{b_2^2} - \frac{a_2b_0}{b_2}. \qquad \text{and} \qquad M = a_1 + \frac{4a_3b_1^2}{b_2^2} - \frac{a_3b_0}{b_2} - \frac{2a_2b_1}{b_2}. \qquad \text{(A.30.4)}$$

The pressure integral of the last term in Eqn. (A.30.3) is well known (see for example section 2.103 of Gradshteyn and Ryzhik (1980)) and is dependent on the sign of the discriminant of the denominator. In our case it can be shown that $b_1^2 > b_0b_2$ over the domain of the "funnel" and also that both b_0 and b_1 are positive, while b_2 is negative and bounded away from zero. The indefinite integral, with respect to sea pressure measured in Pa, of the last term in Eqn. (A.30.3) is (with $N^* = 10^4 N$ and $M^* = 10^4 M$)

$$\int \frac{N + Mp}{b_0 + 2b_1p + b_2p^2}\, dP' = \frac{M^*}{2b_2}\ln\left|b_0 + 2b_1p + b_2p^2\right| + \frac{N^*b_2 - M^*b_1}{2b_2\sqrt{b_1^2 - b_0b_2}}\ln\left|\frac{b_2p + b_1 - \sqrt{b_1^2 - b_0b_2}}{b_2p + b_1 + \sqrt{b_1^2 - b_0b_2}}\right|, \qquad \text{(A.30.5)}$$

The enthalpy $\hat{h}^{25}(S_A,\Theta,p)$ is the definite integral of Eqn. (A.30.3) from P_0 to P, plus $c_p^0\Theta$, being the value of enthalpy at P_0 (i. e. at $p = 0\ \text{dbar}$). Hence the full expression for $\hat{h}^{25}(S_A,\Theta,p)$ is (with $A = b_1 - \sqrt{b_1^2 - b_0b_2}$ and $B = b_1 + \sqrt{b_1^2 - b_0b_2}$)

$$\begin{aligned}\hat{h}^{25}(S_A,\Theta,p) &= c_p^0\Theta + 10^4\left(\frac{a_2}{b_2} - \frac{2a_3b_1}{b_2^2}\right)p + 10^4\frac{a_3}{2b_2}p^2 \\ &\quad + \frac{M^*}{2b_2}\ln\left(1 + \frac{2b_1}{b_0}p + \frac{b_2}{b_0}p^2\right) + \frac{N^* - \frac{b_1}{b_2}M^*}{(B-A)}\ln\left(1 + p\,\frac{b_2}{A}\frac{(B-A)}{(B+b_2p)}\right).\end{aligned} \qquad \text{(A.30.6)}$$

The factor of 10^4 that appears here and in N^* and M^* effectively serves to convert the units of the integration variable from dbar to Pa so that $\hat{h}^{25}(S_A,\Theta,p)$ has units of J kg^{-1}. In these equations S_A is in g kg^{-1}, Θ in °C and p is in dbar. The arguments of the two natural logarithms in Eqn. (A.30.6) are always greater than 1, and in fact they are between 1 and 1.2 even for p as large as 10^4 dbar (note that both b_2 and A are negative). Also, when the enthalpy difference at the same values of S_A and Θ but at different pressures (see Eqn. (3.32.2)) is evaluated using Eqn. (A.30.6), the expression can also be arranged to contain only two logarithm terms.

Following Young (2010), the difference between specific enthalpy and $c_p^0\Theta$ may be called "dynamic enthalpy" and can be readily calculated from Eqn. (A.30.6), recognizing that this equation is based on the computationally efficient 25-term expression for density of McDougall *et al*. (2010b) rather than being evaluated from the full TEOS-10 Gibbs function. Similarly, the partial derivatives of $\hat{h}^{25}(S_A,\Theta,p)$ with respect to Absolute Salinity S_A and with respect to Conservative Temperature Θ can be calculated either by algebraic differentiation of Eqn. (A.30.6) or by first algebraically differentiating Eqn. (A.30.1) and then numerically integrating this expression with respect to pressure (this second procedure is motivated by taking the appropriate S_A or Θ derivatives of Eqn. (3.2.1); see Eqns. (A.18.4) and (A.18.5)).

An expression $\tilde{h}^{25}(S_A,\theta,p)$ for enthalpy as a function of potential temperature θ can be found in a similar manner to that outlined above, but with the coefficients of the 25-term rational-function expression for density now being taken from Table K.2, and with the first term being expressed as the exact polynomial expression for $\tilde{h}(S_A,\theta,0)$ instead of as $c_p^0\Theta$.