Notes on the GSW function gsw_geostrophic velocity

This function calculates the difference between the geostrophic velocity at pressure p and that at the sea surface. The general expression for this velocity difference is

$$\mathbf{k}\times\nabla_{\mathrm{surf}}\vartheta \;=\; f\left(\mathbf{v}-\mathbf{v}_0\right),$$

where f is the Coriolis parameter and ϑ is the geostrophic streamfunction in a particular surface. Note that the geostrophic streamfunction should be carefully chosen to be appropriate for the surface in which it is being used.

Below we reproduce sections 3.27 – 3.30 of the TEOS-10 Manual (IOC *et al.* (2010)) which describe four different types of geostrophic streamfunction. Dynamic height anomaly is the geostrophic streamfunction for use in an isobaric surface, while the Montgomery geostrophic streamfunction is designed for use in a surface of constant specific volume anomaly. The remaining choices of geostrophic streamfunction, the Cunningham and the McDougall-Klocker geostrophic streamfunctions are designed for use in some types of "density" surfaces, and are not exact geostrophic streamfunctions in any particular type of surface.

3.27 Dynamic height anomaly

The dynamic height anomaly Ψ, given by the vertical integral

$$\Psi \;=\; -\int_{P_0}^{P}\delta\left(S_{\mathrm{A}}\left[p'\right],t\left[p'\right],p'\right)dP', \qquad (3.27.1)$$

is the geostrophic streamfunction for the flow at pressure P with respect to the flow at the sea surface and δ is the specific volume anomaly. Thus the two-dimensional gradient of Ψ in the P pressure surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ at P and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k}\times\nabla_P\Psi \;=\; f\mathbf{v}-f\mathbf{v}_0. \qquad (3.27.2)$$

The definition Eqn. (3.27.1) of dynamic height anomaly applies to all choices of the reference values $\tilde{S}_{\mathrm{A}}$ and $\hat{t}$, $\tilde{\theta}$ or $\hat{\Theta}$ in the definition Eqns. (3.7.1 – 3.7.4) of the specific volume anomaly δ. Also, δ in Eqn. (3.27.1) can be replaced with specific volume v without affecting the isobaric gradient of the resulting streamfunction. That is, this substitution does not affect Eqn. (3.27.2) because the additional term is a function only of pressure. Traditionally it was important to use specific volume anomaly in preference to specific volume as it was more accurate with computer code which worked with single-precision variables. Since computers now regularly employ double-precision, this issue has been overcome and consequently either δ or v can be used in the integrand of Eqn. (3.27.1), so making it either the "dynamic height anomaly" or the "dynamic height". As in the case of Eqn. (3.24.2), so also the dynamic height anomaly Eqn. (3.27.1) has not assumed that the gravitational acceleration is constant and so Eqn. (3.27.2) applies even when the gravitational acceleration is taken to vary in the vertical.

The dynamic height anomaly Ψ should be quoted in units of $\mathrm{m^2\,s^{-2}}$. These are the units in which the GSW library (appendix N) outputs dynamic height anomaly in the function gsw_geo_strf_dyn_height. Note that the integration in Eqn. (3.27.1) of specific volume anomaly with pressure in dbar would yield dynamic height anomaly in units of $\mathrm{m^3\,kg^{-1}dbar}$, and the use of these units in Eqn. (3.27.2) would not give the resultant horizontal gradient in the usual units, being the product of the Coriolis parameter (units of $\mathrm{s^{-1}}$) and the velocity (units of $\mathrm{m\,s^{-1}}$). This is the reason why the pressure integration is done with pressure in Pa and dynamic height anomaly is output in $\mathrm{m^2\,s^{-2}}$.

3.28 Montgomery geostrophic streamfunction

The Montgomery "acceleration potential" π defined by

$$\pi = (P - P_0)\delta - \int_{P_0}^{P} \delta\left(S_A[p'], t[p'], p'\right) dP' \qquad (3.28.1)$$

is the geostrophic streamfunction for the flow in the specific volume anomaly surface $\delta(S_A, t, p) = \delta_1$ relative to the flow at $P = P_0$ (that is, at $p = 0\ \text{dbar}$). Thus the two-dimensional gradient of π in the δ_1 specific volume anomaly surface is simply related to the difference between the horizontal geostrophic velocity $\mathbf{v}$ in the $\delta = \delta_1$ surface and at the sea surface $\mathbf{v}_0$ according to

$$\mathbf{k} \times \nabla_{\delta_1} \pi = f\mathbf{v} - f\mathbf{v}_0 \qquad \text{or} \qquad \nabla_{\delta_1} \pi = -\mathbf{k} \times \left(f\mathbf{v} - f\mathbf{v}_0 \right). \qquad (3.28.2)$$

The definition, Eqn. (3.28.1), of the Montgomery geostrophic streamfunction applies to all choices of the reference values $\hat{\hat{S}}_A$ and $\hat{\hat{t}}$ in the definition, Eqn. (3.7.2), of the specific volume anomaly δ. By carefully choosing these reference values the specific volume anomaly surface can be made to closely approximate the neutral tangent plane (McDougall and Jackett (2007)).

It is not uncommon to read of authors using the Montgomery geostrophic streamfunction, Eqn. (3.28.1), as a geostrophic streamfunction in surfaces other than specific volume anomaly surfaces. This incurs errors that should be recognized. For example, the gradient of the Montgomery geostrophic streamfunction, Eqn. (3.28.1), in a neutral tangent plane becomes (instead of Eqn. (3.28.2) in the $\delta = \delta_1$ surface)

$$\nabla_n \pi = -\mathbf{k} \times \left(f\mathbf{v} - f\mathbf{v}_0 \right) + (P - P_0) \nabla_n \delta, \qquad (3.28.3)$$

where the last term represents an error arising from using the Montgomery streamfunction in a surface other than the surface for which it was derived.

Zhang and Hogg (1992) subtracted an arbitrary pressure offset, $(\bar{P} - P_0)$, from $(P - P_0)$ in the first term in Eqn. (3.28.1), so defining the modified Montgomery streamfunction

$$\pi^{\text{Z-H}} = (P - \bar{P})\delta - \int_{P_0}^{P} \delta\left(S_A[p'], t[p'], p'\right) dP'. \qquad (3.28.4)$$

The gradient of $\pi^{\text{Z-H}}$ in a neutral tangent plane becomes

$$\nabla_n \pi^{\text{Z-H}} = -\mathbf{k} \times \left(f\mathbf{v} - f\mathbf{v}_0 \right) + (P - \bar{P}) \nabla_n \delta, \qquad (3.28.5)$$

where the last term can be made significantly smaller than the corresponding term in Eqn. (3.28.3) by choosing the constant pressure $\bar{P}$ to be close to the average pressure on the surface.

This term can be further minimized by suitably choosing the constant reference values $\tilde{\tilde{S}}_A$ and $\tilde{\tilde{\Theta}}$ in the definition, Eqn. (3.7.3), of specific volume anomaly $\tilde{\tilde{\delta}}$ so that this surface more closely approximates the neutral tangent plane (McDougall (1989)). This improvement is available because it can be shown that

$$\rho \nabla_n \tilde{\tilde{\delta}} = -\left[\kappa\left(S_A, \Theta, p\right) - \kappa\left(\tilde{\tilde{S}}_A, \tilde{\tilde{\Theta}}, p\right) \right] \nabla_n P \approx T_b^{\Theta} \left(\Theta - \tilde{\tilde{\Theta}} \right) \nabla_n P. \qquad (3.28.6)$$

The last term in Eqn. (3.28.5) is then approximately

$$(P - \bar{P}) \nabla_n \tilde{\tilde{\delta}} \approx \tfrac{1}{2} \rho^{-1} T_b^{\Theta} \left(\Theta - \tilde{\tilde{\Theta}} \right) \nabla_n (P - \bar{P})^2 \qquad (3.28.7)$$

and hence suitable choices of $\bar{P}$, $\tilde{\tilde{S}}_A$ and $\tilde{\tilde{\Theta}}$ can reduce the last term in Eqn. (3.28.5) that represents the error in interpreting the Montgomery geostrophic streamfunction, Eqn. (3.28.4), as the geostrophic streamfunction in a surface that is more neutral than a specific volume anomaly surface.

The Montgomery geostrophic streamfunction should be quoted in units of $\text{m}^2\ \text{s}^{-2}$. These are the units in which the GSW library (appendix N) outputs the Montgomery geostrophic streamfunction in the function gsw_geo_strf_Montgomery.

3.29 Cunningham geostrophic streamfunction

Cunningham (2000) and Alderson and Killworth (2005), following Saunders (1995) and Killworth (1986), suggested that a suitable streamfunction on a density surface in a compressible ocean would be the difference between the Bernoulli function $\mathscr{B}$ and potential enthalpy h^0. Since the kinetic energy per unit mass, $0.5\mathbf{u}\cdot\mathbf{u}$, is a tiny component of the Bernoulli function, it was ignored and Cunningham (2000) essentially proposed the streamfunction $\Pi+\Phi^0$ (see his equation (12)), where

$$\begin{aligned}\Pi \ &\equiv \mathscr{B} - h^0 - \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} - \Phi^0 \\ &= h - h^0 + \Phi - \Phi^0 \\ &= h(S_\mathrm{A},\Theta,p) - h(S_\mathrm{A},\Theta,0) - \int_{P_0}^{P} \hat{v}\big(S_\mathrm{A}(p'),\Theta(p'),p'\big)\,dP'.\end{aligned} \qquad (3.29.1)$$

The last line of this equation has used the hydrostatic equation $P_z = -g\rho$ to express $\Phi \approx gz$ in terms of the vertical pressure integral of specific volume and the height of the sea surface where the geopotential is Φ^0.

The definition of potential enthalpy, Eqn. (3.2.1), is used to rewrite the last line of Eqn. (3.29.1), showing that Cunningham's Π is also equal to

$$\Pi \ = \ -\int_{P_0}^{P} \hat{v}\big(S_\mathrm{A}(p'),\Theta(p'),p'\big) - \hat{v}\big(S_\mathrm{A},\Theta,p'\big)\,dP'. \qquad (3.29.2)$$

In this form it appears very similar to the expression, Eqn. (3.27.1), for dynamic height anomaly, the only difference being that in Eqn. (3.27.1) the pressure-independent values of Absolute Salinity and Conservative Temperature were S_SO and $0°\mathrm{C}$ whereas here they are the local values on the surface, S_A and Θ. While these local values of Absolute Salinity and Conservative Temperature are constant during the pressure integral in Eqn. (3.29.2), they do vary with latitude and longitude along any "density" surface.

The gradient of Π along the neutral tangent plane is

$$\nabla_n\Pi \ \approx \ \left\{\tfrac{1}{\rho}\nabla_z P - \nabla\Phi_0\right\} - \tfrac{1}{2}\rho^{-1}T_\mathrm{b}^\Theta\left(P - P_0\right)^2\nabla_n\Theta, \qquad (3.29.3)$$

(from McDougall and Klocker (2010)) so that the error in $\nabla_n\Pi$ in using Π as the geostrophic streamfunction is approximately $-\tfrac{1}{2}\rho^{-1}T_\mathrm{b}^\Theta\left(P - P_0\right)^2\nabla_n\Theta$. When using the Cunningham streamfunction Π in a potential density surface, the error in $\nabla_\sigma\Pi$ is approximately $-\tfrac{1}{2}\rho^{-1}T_\mathrm{b}^\Theta\left(P-P_0\right)\left(2P_\mathrm{r}-P-P_0\right)\nabla_\sigma\Theta$. The Cunningham geostrophic streamfunction should be quoted in units of $\text{m}^2\ \text{s}^{-2}$ and is available in the GSW software library (appendix N) as the function gsw_geo_strf_Cunningham.

3.30 Geostrophic streamfunction in an approximately neutral surface

In order to evaluate a relatively accurate expression for the geostrophic streamfunction in an approximately neutral surface (such as an ω-surface of Klocker *et al.* (2009a,b) or a Neutral Density surface of Jackett and McDougall (1997)) a suitable reference seawater parcel $\left(\tilde{\tilde{S}}_\mathrm{A},\tilde{\tilde{\Theta}},\tilde{\tilde{p}}\right)$ is selected from the approximately neutral surface that one is considering, and the specific volume anomaly $\tilde{\tilde{\delta}}$ is defined as in (3.7.3) above. The approximate geostrophic streamfunction is given by (from McDougall and Klocker (2010))

$$\varphi^{\mathrm{n}}\left(S_{\mathrm{A}},\Theta,p\right) \;=\; \tfrac{1}{2}\left(P-\tilde{\tilde{P}}\right)\tilde{\tilde{\delta}}\left(S_{\mathrm{A}},\Theta,p\right) \;-\; \tfrac{1}{12}\rho^{-1}T_{\mathrm{b}}^{\Theta}\left(\Theta-\tilde{\tilde{\Theta}}\right)\left(P-\tilde{\tilde{P}}\right)^{2} \;-\; \int_{P_0}^{P}\tilde{\tilde{\delta}}\,dP'. \qquad (3.30.1)$$

This expression is very accurate when the variation of conservative temperature with pressure along the approximately neutral surface is either linear or quadratic. That is, in these situations $\nabla_n\varphi^{\mathrm{n}} \approx \frac{1}{\rho}\nabla_z P - \nabla\Phi_0 = -\mathbf{k}\times\left(f\mathbf{v} - f\mathbf{v}_0\right)$ to a very good approximation. In Eqn. (3.30.1) $\rho^{-1}T_{\mathrm{b}}^{\Theta}$ is taken to be the constant value $2.7x10^{-15}\,\mathrm{K}^{-1}(\mathrm{Pa})^{-2}\mathrm{m}^2\mathrm{s}^{-2}$. This McDougall-Klocker geostrophic streamfunction is available from the GSW software as the function gsw_geo_strf_McD_Klocker.