3.9 Cabbeling coefficient

The cabbeling coefficient quantifies the rate at which dianeutral advection occurs as a result of mixing of heat and salt along the neutral tangent plane. With respect to potential temperature θ the cabbeling coefficient is (McDougall (1987b))

$$C_{\mathrm{b}}^{\theta} = C_{\mathrm{b}}^{\theta}\left(S_{\mathrm{A}}, t\, p\right) = \left.\frac{\partial \alpha^{\theta}}{\partial \theta}\right|_{S_{\mathrm{A}}, p} + 2\frac{\alpha^{\theta}}{\beta^{\theta}}\left.\frac{\partial \alpha^{\theta}}{\partial S_{\mathrm{A}}}\right|_{\theta, p} - \left(\frac{\alpha^{\theta}}{\beta^{\theta}}\right)^{2}\left.\frac{\partial \beta^{\theta}}{\partial S_{\mathrm{A}}}\right|_{\theta, p}. \qquad (3.9.1)$$

This expression for the cabbeling coefficient is most readily evaluated by differentiating an expression for density expressed as a function of potential temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \tilde{\rho}\left(S_{\mathrm{A}}, \theta, p\right)$.

With respect to Conservative Temperature Θ the cabbeling coefficient is

$$C_{\mathrm{b}}^{\Theta} = C_{\mathrm{b}}^{\Theta}\left(S_{\mathrm{A}}, t\, p\right) = \left.\frac{\partial \alpha^{\Theta}}{\partial \Theta}\right|_{S_{\mathrm{A}}, p} + 2\frac{\alpha^{\Theta}}{\beta^{\Theta}}\left.\frac{\partial \alpha^{\Theta}}{\partial S_{\mathrm{A}}}\right|_{\Theta, p} - \left(\frac{\alpha^{\Theta}}{\beta^{\Theta}}\right)^{2}\left.\frac{\partial \beta^{\Theta}}{\partial S_{\mathrm{A}}}\right|_{\Theta, p}. \qquad (3.9.2)$$

This expression for the cabbeling coefficient is most readily evaluated by differentiating an expression for density expressed as a function of Conservative Temperature rather than *in situ* temperature, that is, with density expressed in the functional form $\rho = \hat{\rho}\left(S_{\mathrm{A}}, \Theta, p\right)$.

The cabbeling dianeutral advection associated with the lateral mixing of heat and salt along neutral tangent planes is given by $e^{\mathrm{Cab}} = -gN^{-2}K\,C_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n\Theta$ (or less accurately by $e^{\mathrm{Cab}} \approx -gN^{-2}K\,C_{\mathrm{b}}^{\theta}\nabla_n\theta\cdot\nabla_n\theta$) where $\nabla_n\theta$ and $\nabla_n\Theta$ are the two-dimensional gradients of either potential temperature or Conservative Temperature along the neutral tangent plane and K is the epineutral diffusion coefficient. The cabbeling dianeutral advection is proportional to the mesoscale eddy flux of "heat" along the neutral tangent plane, $-K\nabla_n\Theta$, and is independent of the amount of small-scale (dianeutral) turbulent mixing and hence is also independent of the dissipation of mechanical energy (Klocker and McDougall (2010a)). It is shown in appendix A.14 that $C_{\mathrm{b}}^{\theta}\nabla_n\theta\cdot\nabla_n\theta \neq C_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n\Theta$ so that the estimate of the cabbeling dianeutral advection is different when calculated using potential temperature than when using Conservative Temperature. The estimate using potential temperature is slightly less accurate because of the non-conservative nature of potential temperature.

When the cabbeling and thermobaricity processes are analyzed by considering the mixing of two fluid parcels one finds that the density change is proportional to the square of the property (Θ and/or p) contrasts between the two fluid parcels (for the cabbeling case, see Eqn. (A.19.4) in appendix A.19). This leads to the thought that if an ocean front is split up into a series of many smaller fronts then the effects of cabbeling and thermobaricity might be reduced by perhaps the square of the number of such fronts. This is not the case. Rather, the total dianeutral transport across a frontal region depends on the product of the lateral flux of heat passing through the front and the contrast in temperature and/or pressure across the front, but is independent of the sharpness of the front (Klocker and McDougall (2010a)). This can be understood by noting from above that the dianeutral velocity due to cabbeling, $e^{\mathrm{Cab}} = -gN^{-2}K\,C_{\mathrm{b}}^{\Theta}\nabla_n\Theta\cdot\nabla_n\Theta$, is proportional to the scalar product of the epineutral flux of heat $-c_p^0 K\nabla_n\Theta$ and the epineutral temperature gradient $\nabla_n\Theta$. Spatially integrating this product over the area of the frontal region, one finds that the total dianeutral transport is proportional to the lateral heat flux times the difference in temperature across the frontal region (in the case of cabbeling) or the difference in pressure across the frontal region (in the case of thermobaricity).

In both the SIA and GSW software libraries the cabbeling parameter is output in units of K^{-2}. Expressions for C_{b}^{θ} and C_{b}^{Θ} in terms of enthalpy in the functional forms $\tilde{h}\left(S_{\mathrm{A}}, \theta, p\right)$ and $\hat{h}\left(S_{\mathrm{A}}, \Theta, p\right)$ can be found in appendix P.

A.14 Advective and diffusive "heat" fluxes

In section 3.23 and appendices A.8 and A.13 the First Law of Thermodynamics is shown to be practically equivalent to the conservation equation (A.21.15) for Conservative Temperature Θ. We have emphasized that this means that the advection of "heat" is very accurately given as the advection of $c_p^0\Theta$. In this way $c_p^0\Theta$ can be regarded as the "heat content" per unit mass of seawater and the error involved with making this association is approximately 1% of the error in assuming that either $c_p^0\theta$ or $c_p\left(S_{\mathrm{A}},\theta,0\,\mathrm{dbar}\right)\theta$ is the "heat content" per unit mass of seawater (see also appendix A.21 for a discussion of this point).

The conservative form (A.21.15) implies that the turbulent diffusive flux of heat should be directed down the mean gradient of Conservative Temperature rather than down the mean gradient of potential temperature. In this appendix we quantify the difference between these mean temperature gradients.

Consider first the respective temperature gradients along the neutral tangent plane. From Eqn. (3.11.2) we find that

$$\left(\alpha^{\theta}/\beta^{\theta}\right)\nabla_n\theta = \nabla_n S_{\mathrm{A}} = \left(\alpha^{\Theta}/\beta^{\Theta}\right)\nabla_n\Theta, \qquad \text{(A.14.1)}$$

so that the epineutral gradients of θ and Θ are related by the ratios of their respective thermal expansion and saline contraction coefficients, namely

$$\nabla_n\theta = \frac{\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\left(\alpha^{\theta}/\beta^{\theta}\right)}\nabla_n\Theta. \qquad \text{(A.14.2)}$$

This proportionality factor between the parallel two-dimensional vectors $\nabla_n\theta$ and $\nabla_n\Theta$ is readily calculated and illustrated graphically. Before doing so we note two other equivalent expressions for this proportionality factor.

The epineutral gradients of θ, Θ and S_{A} are related by (using $\theta = \hat{\theta}\left(S_{\mathrm{A}},\Theta\right)$)

$$\nabla_n\theta = \hat{\theta}_{\Theta}\nabla_n\Theta + \hat{\theta}_{S_{\mathrm{A}}}\nabla_n S_{\mathrm{A}}, \qquad \text{(A.14.3)}$$

and using the neutral relationship $\nabla_n S_{\mathrm{A}} = \left(\alpha^{\Theta}/\beta^{\Theta}\right)\nabla_n\Theta$ we find

$$\nabla_n\theta = \left(\hat{\theta}_{\Theta} + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_{\mathrm{A}}}\right)\nabla_n\Theta. \qquad \text{(A.14.4)}$$

Also, in section 3.13 we found that $T_b^{\theta}\nabla_n\theta = T_b^{\Theta}\nabla_n\Theta$, so that we can write the equivalent expressions

$$\frac{\left|\nabla_n\theta\right|}{\left|\nabla_n\Theta\right|} = \frac{\left(\alpha^{\Theta}/\beta^{\Theta}\right)}{\left(\alpha^{\theta}/\beta^{\theta}\right)} = \frac{T_b^{\Theta}}{T_b^{\theta}} = \hat{\theta}_{\Theta} + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_{\mathrm{A}}}, \qquad \text{(A.14.5)}$$

and it can be shown that $\alpha^{\Theta}/\alpha^{\theta} = \hat{\theta}_{\Theta}$ and $\beta^{\theta}/\beta^{\Theta} = \left(1 + \left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_{\mathrm{A}}}/\hat{\theta}_{\Theta}\right)$, that is, $\beta^{\theta} = \beta^{\Theta} + \alpha^{\Theta}\,\hat{\theta}_{S_{\mathrm{A}}}/\hat{\theta}_{\Theta}$. The partial derivatives $\hat{\theta}_{\Theta}$ and $\hat{\theta}_{S_{\mathrm{A}}}$ in the last part of Eqn. (A.14.5) are both independent of pressure while $\alpha^{\Theta}/\beta^{\Theta}$ is a function of pressure. This ratio, Eqn. (A.14.5), of the epineutral gradients of θ and Θ is shown in Figure A.14.1 at $p = 0$, indicating that the epineutral gradient of potential temperature is sometimes more that 1% different to that of Conservative Temperature. This ratio $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ is only a weak function of pressure. This ratio, $\left|\nabla_n\theta\right|/\left|\nabla_n\Theta\right|$ (i.e. Eqn. (A.14.5)), is available in the GSW Oceanographic Toolbox as function **gsw_ntp_pt_vs_CT_ratio_CT25**.

Similarly to Eqn. (A.14.3), the vertical gradients are related by

$$\theta_z = \hat{\theta}_{\Theta}\Theta_z + \hat{\theta}_{S_{\mathrm{A}}}S_{\mathrm{A}_z}, \qquad \text{(A.14.6)}$$

and using the definition, Eqn. (3.15.1), of the stability ratio we find that

$$\frac{\theta_z}{\Theta_z} = \hat{\theta}_{\Theta} + R_{\rho}^{-1}\left[\alpha^{\Theta}/\beta^{\Theta}\right]\hat{\theta}_{S_{\mathrm{A}}}. \qquad \text{(A.14.7)}$$

For values of the stability ratio R_ρ close to unity, the ratio θ_z / Θ_z is close to the values of $|\nabla_n \theta| / |\nabla_n \Theta|$ shown in Figure A.14.1. For other values of R_ρ, Eqn. (A.14.7) can be calculated and plotted.

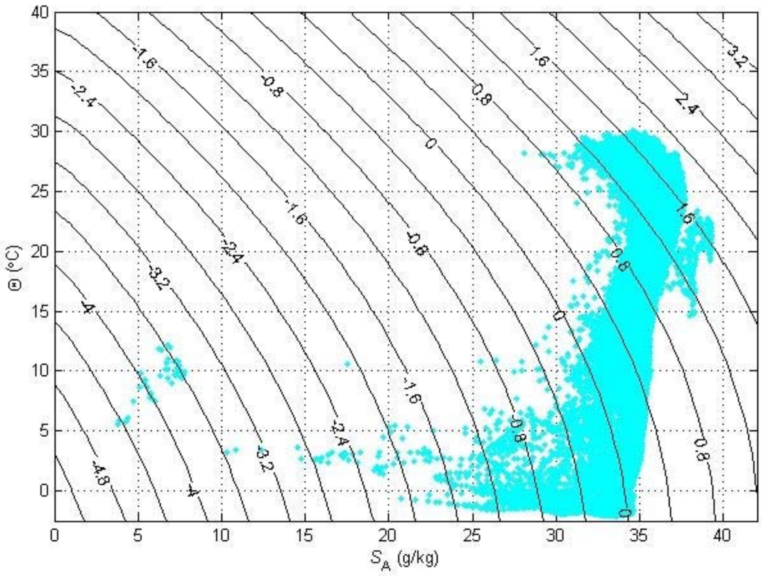

Figure A.14.1. Contours of $\left(|\nabla_n \theta| / |\nabla_n \Theta| \; - 1 \right) \times 100\%$ at $p = 0$, showing the percentage difference between the epineutral gradients of θ and Θ. The blue dots are from the ocean atlas of Gouretski and Koltermann (2004) at $p = 0$.

As noted in section 3.8 the dianeutral advection of thermobaricity is the same when quantified in terms of potential temperature as when done in terms of Conservative Temperature. The same is not true of the dianeutral velocity caused by cabbeling. The ratio of the cabbeling dianeutral velocity calculated using potential temperature to that using Conservative Temperature is given by $\left(C_b^\theta \nabla_n \theta \cdot \nabla_n \theta \right) / \left(C_b^\Theta \nabla_n \Theta \cdot \nabla_n \Theta \right)$ (see section 3.9) which can be expressed as

$$\frac{C_b^\theta |\nabla_n \theta|^2}{C_b^\Theta |\nabla_n \Theta|^2} = \frac{C_b^\theta}{C_b^\Theta} \frac{\left(\alpha^\Theta / \beta^\Theta \right)^2}{\left(\alpha^\theta / \beta^\theta \right)^2} = \frac{C_b^\theta}{C_b^\Theta} \left(\frac{T_b^\Theta}{T_b^\theta} \right)^2 = \frac{C_b^\theta}{C_b^\Theta} \left(\hat{\theta}_\Theta + \left[\alpha^\Theta / \beta^\Theta \right] \hat{\theta}_{S_A} \right)^2 , \qquad \text{(A.14.8)}$$

and this is contoured in Fig. A.14.2. While the ratio of Eqn. (A.14.8) is not exactly unity, it varies relatively little in the oceanographic range, indicating that the dianeutral advection due to cabbeling estimated using θ or Θ are within half a percent of each other at $p = 0$.

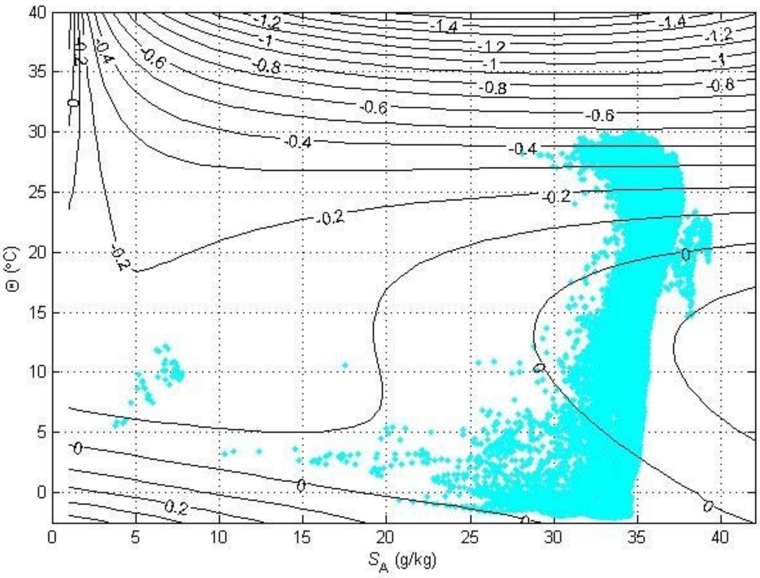

Figure A.14.2. Contours of the percentage difference of $\left(C_{\mathrm{b}}^{\theta}\left|\nabla_n\theta\right|^2\right)\Big/\left(C_{\mathrm{b}}^{\Theta}\left|\nabla_n\Theta\right|^2\right)$ from unity at $p = 0$ dbar.