A.12 Differential relationships between η, θ, Θ and S_A

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\mathrm{d}\eta+\mu(p)\mathrm{d}S_A &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\,\mathrm{d}\theta\;+\;\big[\mu(p)-(T_0+t)\,\mu_T(0)\big]\mathrm{d}S_A \\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\,\mathrm{d}\Theta\;+\;\left[\mu(p)-\frac{(T_0+t)}{(T_0+\theta)}\mu(0)\right]\mathrm{d}S_A\,.\end{aligned} \qquad \text{(A.12.1)}$$

The quantity $\mu(p)\mathrm{d}S_A$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\mathrm{d}\eta \;=\; c_p(0)\,\mathrm{d}\theta-(T_0+\theta)\,\mu_T(0)\,\mathrm{d}S_A \;=\; c_p^0\,\mathrm{d}\Theta-\mu(0)\,\mathrm{d}S_A\,. \qquad \text{(A.12.2)}$$

From this follows all the following partial derivatives between η, θ, Θ and S_A,

$$\Theta_\theta\big|_{S_A}=c_p(S_A,\theta,0)/c_p^0,\qquad \Theta_{S_A}\big|_\theta=\big[\mu(S_A,\theta,0)-(T_0+\theta)\,\mu_T(S_A,\theta,0)\big]/c_p^0, \qquad \text{(A.12.3)}$$

$$\Theta_\eta\big|_{S_A}=(T_0+\theta)/c_p^0,\qquad \Theta_{S_A}\big|_\eta=\mu(S_A,\theta,0)/c_p^0, \qquad \text{(A.12.4)}$$

$$\theta_\eta\big|_{S_A}=(T_0+\theta)/c_p(S_A,\theta,0),\qquad \theta_{S_A}\big|_\eta=(T_0+\theta)\,\mu_T(S_A,\theta,0)/c_p(S_A,\theta,0), \qquad \text{(A.12.5)}$$

$$\theta_\Theta\big|_{S_A}=c_p^0/c_p(S_A,\theta,0),\qquad \theta_{S_A}\big|_\Theta=-\big[\mu(S_A,\theta,0)-(T_0+\theta)\,\mu_T(S_A,\theta,0)\big]/c_p(S_A,\theta,0), \qquad \text{(A.12.6)}$$

$$\eta_\theta\big|_{S_A}=c_p(S_A,\theta,0)/(T_0+\theta),\qquad \eta_{S_A}\big|_\theta=-\mu_T(S_A,\theta,0), \qquad \text{(A.12.7)}$$

$$\eta_\Theta\big|_{S_A}=c_p^0/(T_0+\theta),\qquad \eta_{S_A}\big|_\Theta=-\mu(S_A,\theta,0)/(T_0+\theta). \qquad \text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}(S_A,\Theta)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}(S_A,\Theta)$, namely $\hat{\theta}_\Theta$, $\hat{\theta}_{S_A}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_A\Theta}$ and $\hat{\theta}_{S_AS_A}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_\Theta=\frac{1}{\tilde{\Theta}_\theta},\quad \hat{\theta}_{S_A}=-\frac{\tilde{\Theta}_{S_A}}{\tilde{\Theta}_\theta},\quad \hat{\theta}_{\Theta\Theta}=-\frac{\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3},\quad \hat{\theta}_{S_A\Theta}=-\frac{\tilde{\Theta}_{\theta S_A}}{\big(\tilde{\Theta}_\theta\big)^2}+\frac{\tilde{\Theta}_{S_A}\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \qquad \text{(A.12.9a,b,c,d)}$$

$$\text{and}\quad \hat{\theta}_{S_AS_A}=-\frac{\tilde{\Theta}_{S_AS_A}}{\tilde{\Theta}_\theta}+2\,\frac{\tilde{\Theta}_{S_A}}{\tilde{\Theta}_\theta}\frac{\tilde{\Theta}_{\theta S_A}}{\tilde{\Theta}_\theta}-\left(\frac{\tilde{\Theta}_{S_A}}{\tilde{\Theta}_\theta}\right)^2\frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_\theta}, \qquad \text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_\theta$, $\tilde{\Theta}_{S_A}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_A}$ and $\tilde{\Theta}_{S_AS_A}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}(S_A,\theta)$ from the TEOS-10 Gibbs function.

… and an excerpt from appendix P

The partial derivatives with respect to Θ and with respect to θ, both at constant S_A and p, and the partial derivatives with respect to S_A, are related by

$$\left.\frac{\partial}{\partial\Theta}\right|_{S_A,p}=\frac{1}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_A,p},\quad\text{and}\quad \left.\frac{\partial}{\partial S_A}\right|_{\Theta,p}=\left.\frac{\partial}{\partial S_A}\right|_{\theta,p}-\frac{\tilde{\Theta}_{S_A}}{\tilde{\Theta}_\theta}\left.\frac{\partial}{\partial\theta}\right|_{S_A,p}. \qquad \text{(P.13a,b)}$$

Use of these expressions, acting on entropy yields (with $p=0$ everywhere, and using Eqn. (P.7) [or Eqn. (A.12.8b)] and Eqn. (P.8))

$$\hat{\eta}_{\Theta} = \frac{\tilde{\eta}_{\theta}}{\tilde{\Theta}_{\theta}} \equiv \frac{c_p^0}{(T_0 + \theta)}, \quad \hat{\eta}_{\Theta\Theta} = -\frac{1}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0 + \theta)^2}, \quad \hat{\eta}_{S_\mathrm{A}} = -\frac{\tilde{\mu}(S_\mathrm{A}, \theta, 0)}{(T_0 + \theta)}, \qquad \text{(P.14a,b,c)}$$

$$\hat{\eta}_{S_\mathrm{A}\Theta} = \frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0 + \theta)^2}, \quad \text{and} \quad \hat{\eta}_{S_\mathrm{A}S_\mathrm{A}} = -\frac{\tilde{\mu}_{S_\mathrm{A}}(S_\mathrm{A}, \theta, 0)}{(T_0 + \theta)} - \frac{\left(\tilde{\Theta}_{S_\mathrm{A}}\right)^2}{\tilde{\Theta}_{\theta}} \frac{c_p^0}{(T_0 + \theta)^2}, \qquad \text{(P.15a,b)}$$

in terms of the partial derivatives of the exact polynomial expressions (P.11b) and (P.12).