2.18 Thermal expansion coefficients

The thermal expansion coefficient α^t with respect to *in situ* temperature t, is

$$\alpha^t = \alpha^t(S_A,t,p) = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_{S_A,p} = \frac{g_{TP}}{g_P}. \tag{2.18.1}$$

The thermal expansion coefficient α^θ with respect to potential temperature θ, is (see appendix A.15)

$$\alpha^\theta = \alpha^\theta(S_A,t,p,p_r) = -\frac{1}{\rho}\frac{\partial\rho}{\partial\theta}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial\theta}\bigg|_{S_A,p} = \frac{g_{TP}}{g_P}\frac{g_{TT}(S_A,\theta,p_r)}{g_{TT}}, \tag{2.18.2}$$

where p_r is the reference pressure of the potential temperature. The g_{TT} derivative in the numerator is evaluated at (S_A,θ,p_r) whereas the other derivatives are all evaluated at (S_A,t,p).

The thermal expansion coefficient α^Θ with respect to Conservative Temperature Θ, is (see appendix A.15)

$$\alpha^\Theta = \alpha^\Theta(S_A,t,p) = -\frac{1}{\rho}\frac{\partial\rho}{\partial\Theta}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial\Theta}\bigg|_{S_A,p} = -\frac{g_{TP}}{g_P}\frac{c_p^0}{(T_0+\theta)\,g_{TT}}. \tag{2.18.3}$$

Note that Conservative Temperature Θ is defined only with respect to a reference pressure of 0 dbar so that the θ in Eqn. (2.18.3) is the potential temperature with $p_r = 0$ dbar. All the derivatives on the right-hand side of Eqn. (2.18.3) are evaluated at (S_A,t,p). The constant c_p^0 is defined in Eqn. (3.3.3) below.

A.15 Derivation of the expressions for α^θ, β^θ, α^Θ and β^Θ

This appendix derives the expressions in Eqns. (2.18.2) – (2.18.3) and (2.19.2) – (2.19.3) for the thermal expansion coefficients α^θ and α^Θ and the haline contraction coefficients β^θ and β^Θ.

In order to derive Eqn. (2.18.2) for α^θ we first need an expression for $\partial\theta/\partial t|_{S_A,p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_A,t,p) = \eta(S_A,\theta[S_A,t,p,p_r],p_r)$ which defines potential temperature, obtaining

$$\frac{\partial\theta}{\partial t}\bigg|_{S_A,p} = \frac{\eta_T(S_A,t,p)}{\eta_T(S_A,\theta,p_r)} = \frac{g_{TT}(S_A,t,p)}{g_{TT}(S_A,\theta,p_r)}. \tag{A.15.1}$$

This is then used to obtain the desired expression Eqn. (2.18.2) for α^θ as follows

$$\alpha^\theta = \frac{1}{v}\frac{\partial v}{\partial\theta}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial t}\bigg|_{S_A,p}\left(\frac{\partial\theta}{\partial t}\bigg|_{S_A,p}\right)^{-1} = \frac{g_{TP}(S_A,t,p)}{g_P(S_A,t,p)}\frac{g_{TT}(S_A,\theta,p_r)}{g_{TT}(S_A,t,p)}. \tag{A.15.2}$$

In order to derive Eqn. (2.18.3) for α^Θ we first need an expression for $\partial\Theta/\partial t|_{S_A,p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_A,t,p) = \hat{\eta}(S_A,\Theta[S_A,t,p])$ obtaining

$$\frac{\partial\Theta}{\partial t}\bigg|_{S_A,p} = \eta_T(S_A,t,p)\frac{\partial\Theta}{\partial\eta}\bigg|_{S_A} = -(T_0+\theta)\,g_{TT}(S_A,t,p)/c_p^0, \tag{A.15.3}$$

where the second part of this equation has used Eqn. (A.12.4) for $\Theta_\eta|_{S_A}$. This is then used to obtain the desired expression Eqn. (2.18.3) for α^Θ as follows

$$\alpha^{\Theta} = \frac{1}{v}\frac{\partial v}{\partial \Theta}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial t}\bigg|_{S_A,p}\left(\frac{\partial \Theta}{\partial t}\bigg|_{S_A,p}\right)^{-1} = -\frac{g_{TP}(S_A,t,p)}{g_P(S_A,t,p)}\frac{c_p^0}{(T_0+\theta)\,g_{TT}(S_A,t,p)}. \qquad \text{(A.15.4)}$$

In order to derive Eqn. (2.19.2) for β^{θ} we first need an expression for $\partial\theta/\partial S_A\big|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta(S_A,t,p) = \eta(S_A,\theta[S_A,t,p,p_r],p_r)$ which defines potential temperature, obtaining

$$\begin{aligned}\frac{\partial\theta}{\partial S_A}\bigg|_{T,p} &= \theta_{\eta}\big|_{S_A}\left[\eta_{S_A}(S_A,t,p) - \eta_{S_A}(S_A,\theta,p_r)\right]\\ &= \frac{(T_0+\theta)}{c_p(S_A,\theta,p_r)}\left[\mu_T(S_A,\theta,p_r) - \mu_T(S_A,t,p)\right]\\ &= \left[g_{S_AT}(S_A,t,p) - g_{S_AT}(S_A,\theta,p_r)\right]\big/ g_{TT}(S_A,\theta,p_r),\end{aligned} \qquad \text{(A.15.5)}$$

where Eqns. (A.12.5) and (A.12.7) have been used with a general reference pressure p_r rather than with $p_r = 0$. By differentiating $\rho = \tilde{\rho}(S_A,\theta[S_A,t,p,p_r],p)$ with respect to Absolute Salinity it can be shown that (Gill (1982), McDougall (1987a))

$$\beta^{\theta} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{\theta,p} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{T,p} + \alpha^{\theta}\frac{\partial\theta}{\partial S_A}\bigg|_{T,p}, \qquad \text{(A.15.6)}$$

and using Eqn. (A.15.5) we arrive at the desired expression Eqn. (2.19.2) for β^{θ}

$$\beta^{\theta} = -\frac{g_{S_AP}(S_A,t,p)}{g_P(S_A,t,p)} + \frac{g_{TP}(S_A,t,p)\left[g_{S_AT}(S_A,t,p) - g_{S_AT}(S_A,\theta,p_r)\right]}{g_P(S_A,t,p)\,g_{TT}(S_A,t,p)}. \qquad \text{(A.15.7)}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.5) and (A.15.7).

In order to derive Eqn. (2.19.3) for β^{Θ} we first need an expression for $\partial\Theta/\partial S_A\big|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta(S_A,t,p) = \hat{\eta}(S_A,\Theta[S_A,t,p])$ obtaining (using Eqns. (A.12.4) and (A.12.8))

$$\begin{aligned}\frac{\partial\Theta}{\partial S_A}\bigg|_{T,p} &= \Theta_{\eta}\big|_{S_A}\left[\eta_{S_A}(S_A,t,p) - \hat{\eta}_{S_A}\big|_{\Theta}\right]\\ &= \left[\mu(S_A,\theta,0) - (T_0+\theta)\,\mu_T(S_A,t,p)\right]\big/ c_p^0\\ &= \left[g_{S_A}(S_A,\theta,0) - (T_0+\theta)\,g_{S_AT}(S_A,t,p)\right]\big/ c_p^0.\end{aligned} \qquad \text{(A.15.8)}$$

Differentiating $\rho = \hat{\rho}(S_A,\Theta[S_A,t,p],p)$ with respect to Absolute Salinity leads to

$$\beta^{\Theta} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{\Theta,p} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{T,p} + \alpha^{\Theta}\frac{\partial\Theta}{\partial S_A}\bigg|_{T,p}, \qquad \text{(A.15.9)}$$

and using Eqn. (A.15.8) we arrive at the desired expression (2.19.3) for β^{Θ} namely

$$\beta^{\Theta} = -\frac{g_{S_AP}(S_A,t,p)}{g_P(S_A,t,p)} + \frac{g_{TP}(S_A,t,p)\left[g_{S_AT}(S_A,t,p) - g_{S_A}(S_A,\theta,0)/(T_0+\theta)\right]}{g_P(S_A,t,p)\,g_{TT}(S_A,t,p)}. \qquad \text{(A.15.10)}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.8) and (A.15.10).