

## 2.19 Saline contraction coefficients

The saline contraction coefficient  $\beta^t$  (sometimes also called the haline contraction coefficient) at constant *in situ* temperature  $t$ , is

$$\beta^t = \beta^t(S_A, t, p) = \left. \frac{1}{\rho} \frac{\partial \rho}{\partial S_A} \right|_{T, p} = - \left. \frac{1}{v} \frac{\partial v}{\partial S_A} \right|_{T, p} = - \frac{g_{S_A P}}{g_P}. \quad (2.19.1)$$

The saline contraction coefficient  $\beta^\theta$  at constant potential temperature  $\theta$ , is (see appendix A.15)

$$\begin{aligned} \beta^\theta = \beta^\theta(S_A, t, p, p_r) &= \left. \frac{1}{\rho} \frac{\partial \rho}{\partial S_A} \right|_{\theta, p} = - \left. \frac{1}{v} \frac{\partial v}{\partial S_A} \right|_{\theta, p} \\ &= \frac{g_{TP} [g_{S_A T} - g_{S_A T}(S_A, \theta, p_r)] - g_{TT} g_{S_A P}}{g_P g_{TT}}, \end{aligned} \quad (2.19.2)$$

where  $p_r$  is the reference pressure of  $\theta$ . One of the  $g_{S_A T}$  derivatives in the numerator is evaluated at  $(S_A, \theta, p_r)$  whereas all the other derivatives are evaluated at  $(S_A, t, p)$ .

The saline contraction coefficient  $\beta^\Theta$  at constant Conservative Temperature  $\Theta$ , is (see appendix A.15)

$$\begin{aligned} \beta^\Theta = \beta^\Theta(S_A, t, p) &= \left. \frac{1}{\rho} \frac{\partial \rho}{\partial S_A} \right|_{\Theta, p} = - \left. \frac{1}{v} \frac{\partial v}{\partial S_A} \right|_{\Theta, p} \\ &= \frac{g_{TP} [g_{S_A T} - (T_0 + \theta)^{-1} g_{S_A}(S_A, \theta, 0)] - g_{TT} g_{S_A P}}{g_P g_{TT}}. \end{aligned} \quad (2.19.3)$$

Note that Conservative Temperature  $\Theta$  is defined only with respect to a reference pressure of 0 dbar as indicated in this equation. The  $g_{S_A}$  derivative in the numerator is evaluated at  $(S_A, \theta, 0)$  whereas all the other derivatives are evaluated at  $(S_A, t, p)$ .

In the SIA computer software (appendix M) all three saline contraction coefficients are produced in units of  $\text{kg kg}^{-1}$  while in the GSW library (appendix N) all three saline contraction coefficients are produced in units of  $\text{kg g}^{-1}$  consistent with the preferred oceanographic unit for  $S_A$  in the GSW library being  $\text{g kg}^{-1}$ .

## A.15 Derivation of the expressions for $\alpha^\theta$ , $\beta^\theta$ , $\alpha^\Theta$ and $\beta^\Theta$

This appendix derives the expressions in Eqns. (2.18.2) – (2.18.3) and (2.19.2) – (2.19.3) for the thermal expansion coefficients  $\alpha^\theta$  and  $\alpha^\Theta$  and the haline contraction coefficients  $\beta^\theta$  and  $\beta^\Theta$ .

In order to derive Eqn. (2.18.2) for  $\alpha^\theta$  we first need an expression for  $\partial\theta/\partial t|_{S_A, p}$ . This is found by differentiating with respect to *in situ* temperature the entropy equality  $\eta(S_A, t, p) = \eta(S_A, \theta[S_A, t, p, p_r], p_r)$  which defines potential temperature, obtaining

$$\left. \frac{\partial \theta}{\partial t} \right|_{S_A, p} = \frac{\eta_T(S_A, t, p)}{\eta_T(S_A, \theta, p_r)} = \frac{g_{TT}(S_A, t, p)}{g_{TT}(S_A, \theta, p_r)}. \quad (A.15.1)$$

This is then used to obtain the desired expression Eqn. (2.18.2) for  $\alpha^\theta$  as follows

$$\alpha^\theta = \left. \frac{1}{v} \frac{\partial v}{\partial \theta} \right|_{S_A, p} = \left. \frac{1}{v} \frac{\partial v}{\partial t} \right|_{S_A, p} \left( \left. \frac{\partial \theta}{\partial t} \right|_{S_A, p} \right)^{-1} = \frac{g_{TP}(S_A, t, p)}{g_P(S_A, t, p)} \frac{g_{TT}(S_A, \theta, p_r)}{g_{TT}(S_A, t, p)}. \quad (A.15.2)$$

In order to derive Eqn. (2.18.3) for  $\alpha^\Theta$  we first need an expression for  $\partial\Theta/\partial t|_{S_A, p}$ . This is found by differentiating with respect to *in situ* temperature the entropy equality  $\eta(S_A, t, p) = \hat{\eta}(S_A, \Theta[S_A, t, p])$  obtaining

$$\left. \frac{\partial \Theta}{\partial t} \right|_{S_A, p} = \eta_T(S_A, t, p) \left. \frac{\partial \Theta}{\partial \eta} \right|_{S_A} = -(T_0 + \theta) g_{TT}(S_A, t, p) / c_p^0, \quad (\text{A.15.3})$$

where the second part of this equation has used Eqn. (A.12.4) for  $\Theta_{\eta}|_{S_A}$ . This is then used to obtain the desired expression Eqn. (2.18.3) for  $\alpha^\ominus$  as follows

$$\alpha^\ominus = \frac{1}{v} \left. \frac{\partial v}{\partial \Theta} \right|_{S_A, p} = \frac{1}{v} \left. \frac{\partial v}{\partial t} \right|_{S_A, p} \left( \left. \frac{\partial \Theta}{\partial t} \right|_{S_A, p} \right)^{-1} = - \frac{g_{TP}(S_A, t, p)}{g_P(S_A, t, p)} \frac{c_p^0}{(T_0 + \theta) g_{TT}(S_A, t, p)}. \quad (\text{A.15.4})$$

In order to derive Eqn. (2.19.2) for  $\beta^\theta$  we first need an expression for  $\partial \theta / \partial S_A|_{T, p}$ . This is found by differentiating with respect to Absolute Salinity the entropy equality  $\eta(S_A, t, p) = \eta(S_A, \theta[S_A, t, p, p_r], p_r)$  which defines potential temperature, obtaining

$$\begin{aligned} \left. \frac{\partial \theta}{\partial S_A} \right|_{T, p} &= \theta_{\eta}|_{S_A} [\eta_{S_A}(S_A, t, p) - \eta_{S_A}(S_A, \theta, p_r)] \\ &= \frac{(T_0 + \theta)}{c_p(S_A, \theta, p_r)} [\mu_T(S_A, \theta, p_r) - \mu_T(S_A, t, p)] \\ &= [g_{S_A T}(S_A, t, p) - g_{S_A T}(S_A, \theta, p_r)] / g_{TT}(S_A, \theta, p_r), \end{aligned} \quad (\text{A.15.5})$$

where Eqns. (A.12.5) and (A.12.7) have been used with a general reference pressure  $p_r$  rather than with  $p_r = 0$ . By differentiating  $\rho = \tilde{\rho}(S_A, \theta[S_A, t, p, p_r], p)$  with respect to Absolute Salinity it can be shown that (Gill (1982), McDougall (1987a))

$$\beta^\theta = \frac{1}{\rho} \left. \frac{\partial \rho}{\partial S_A} \right|_{\theta, p} = \frac{1}{\rho} \left. \frac{\partial \rho}{\partial S_A} \right|_{T, p} + \alpha^\theta \left. \frac{\partial \theta}{\partial S_A} \right|_{T, p}, \quad (\text{A.15.6})$$

and using Eqn. (A.15.5) we arrive at the desired expression Eqn. (2.19.2) for  $\beta^\theta$

$$\beta^\theta = - \frac{g_{S_A P}(S_A, t, p)}{g_P(S_A, t, p)} + \frac{g_{TP}(S_A, t, p) [g_{S_A T}(S_A, t, p) - g_{S_A T}(S_A, \theta, p_r)]}{g_P(S_A, t, p) g_{TT}(S_A, t, p)}. \quad (\text{A.15.7})$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.5) and (A.15.7).

In order to derive Eqn. (2.19.3) for  $\beta^\ominus$  we first need an expression for  $\partial \Theta / \partial S_A|_{T, p}$ . This is found by differentiating with respect to Absolute Salinity the entropy equality  $\eta(S_A, t, p) = \hat{\eta}(S_A, \Theta[S_A, t, p])$  obtaining (using Eqns. (A.12.4) and (A.12.8))

$$\begin{aligned} \left. \frac{\partial \Theta}{\partial S_A} \right|_{T, p} &= \Theta_{\eta}|_{S_A} [\eta_{S_A}(S_A, t, p) - \hat{\eta}_{S_A}|_{\Theta}] \\ &= [\mu(S_A, \theta, 0) - (T_0 + \theta) \mu_T(S_A, t, p)] / c_p^0 \\ &= [g_{S_A}(S_A, \theta, 0) - (T_0 + \theta) g_{S_A T}(S_A, t, p)] / c_p^0. \end{aligned} \quad (\text{A.15.8})$$

Differentiating  $\rho = \hat{\rho}(S_A, \Theta[S_A, t, p], p)$  with respect to Absolute Salinity leads to

$$\beta^\ominus = \frac{1}{\rho} \left. \frac{\partial \rho}{\partial S_A} \right|_{\Theta, p} = \frac{1}{\rho} \left. \frac{\partial \rho}{\partial S_A} \right|_{T, p} + \alpha^\ominus \left. \frac{\partial \Theta}{\partial S_A} \right|_{T, p}, \quad (\text{A.15.9})$$

and using Eqn. (A.15.8) we arrive at the desired expression (2.19.3) for  $\beta^\ominus$  namely

$$\beta^\ominus = - \frac{g_{S_A P}(S_A, t, p)}{g_P(S_A, t, p)} + \frac{g_{TP}(S_A, t, p) [g_{S_A T}(S_A, t, p) - g_{S_A}(S_A, \theta, 0) / (T_0 + \theta)]}{g_P(S_A, t, p) g_{TT}(S_A, t, p)}. \quad (\text{A.15.10})$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.8) and (A.15.10).